

CS103  
FALL 2025

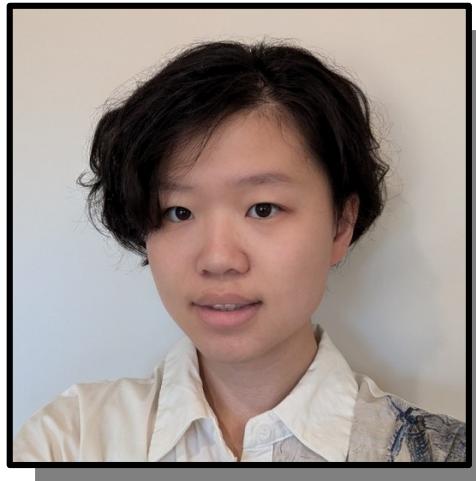


# Lecture 04: First-Order Logic

Part 1 of 2

# Announcements

- Round 2 of problem set matchmaking opens today!
- We're pleased to announce Kaia Li will be our Head TA for the course!



***Kaia Li***  
Head TA

# Recap from Last Time

# Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are as follows:

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*implication (“if  $P$ , then  $Q$ ”)*

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*conjunction (“P and Q”)*

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*truth*

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*negation (“not  $P$ ”)*

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*disjunction (“P or Q”)*

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*falsity*

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*biconditional (“P if and only if Q”)*

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$p$	$q$	$p \rightarrow q$
F	F	—
F	T	—
T	F	—
T	T	—

Answer at  
<https://cs103.stanford.edu/pollev>

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F	F	T	
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Negation of  
 $p \rightarrow q$

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F	F	T	F
F	T	T	F
T	F	F	T
T	T	T	F

Negation of  
 $p \rightarrow q$

# Operator Precedence

- How do we parse this statement?

$$\neg x \rightarrow y \vee z \rightarrow x \vee y \wedge z$$

- Operator precedence for propositional logic:

$\neg$

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$

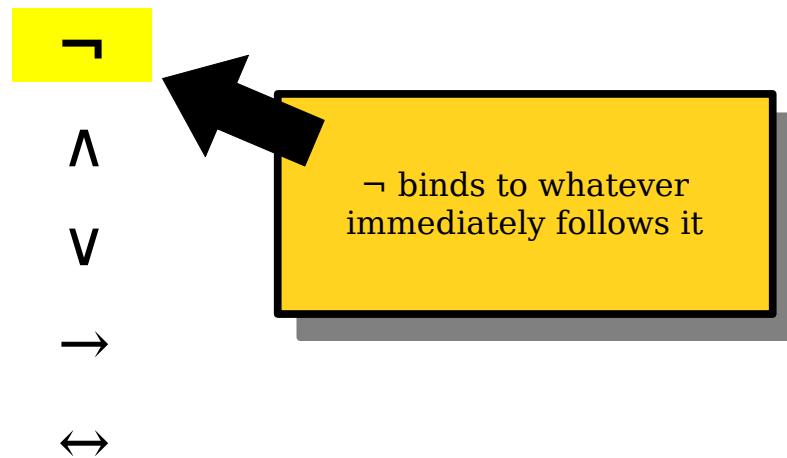
- All operators are right-associative.
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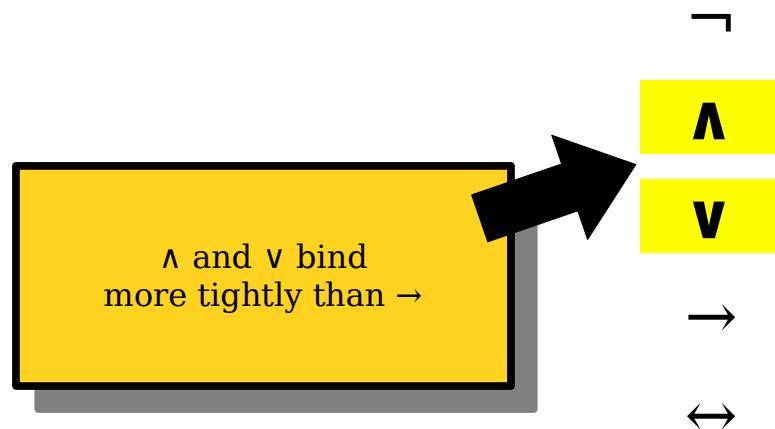
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$$x < 8 \wedge y < 8 \rightarrow x + y \neq 16$$

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“If  $x + y = 16$ , then  $x \geq 8$  or  $y \geq 8$ ”

$$x < 8 \wedge y < 8 \rightarrow x + y \neq 16$$

“If  $x < 8$  and  $y < 8$ , then  $x + y \neq 16$ ”

New Stuff!

# First-Order Logic

# What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about multiple objects.

# Some Examples

*Likes(You, Eggs)  $\wedge$  Likes(You, Tomato)  $\rightarrow$  Likes(You, Shakshuka)*



*Likes(You, Eggs)  $\wedge$  Likes(You, Tomato)  $\rightarrow$  Likes(You, Shakshuka)*

*Learns(You, History)  $\vee$  ForeverRepeats(You, History)*

*In(MyHeart, Havana)  $\wedge$  TookBackTo(Him, Me, EastAtlanta)*

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These blue terms are called *constant symbols*. Unlike propositional variables, they refer to objects, not propositions.

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The red things that look like function calls are called ***predicates***. Predicates take objects as arguments and evaluate to true or false.

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*Learns(You, History)  $\vee$  ForeverRepeats(You, History)*

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What remains are traditional propositional connectives. Because each predicate evaluates to true or false, we can connect the truth values of predicates using normal propositional connectives.

# Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:

*Cute(??)*



# Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:

*Cute(Quokka)*

*ArgueIncessantly(Democrats, Republicans)*

- Applying a predicate to arguments produces a proposition, which is either true or false.
- Typically, when you're working in FOL, you'll have a list of predicates, what they stand for, and how many arguments they take. It'll be given separately than the formulas you write.

# First-Order Formulas

- Formulas in first-order logic can be constructed from predicates applied to objects:

*Cute(a) → Quokka(a) ∨ Kitty(a) ∨ Puppy(a)*

*Succeeds(You) ↔ Practices(You)*

$x < 8 \rightarrow x < 137$

The less-than sign is just another predicate. Binary predicates are sometimes written in *infix notation* this way.

Numbers are not “built in” to first-order logic. They’re constant symbols just like “You” and “a” above.

# Equality

- First-order logic is equipped with a special predicate  $=$  that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as  $\rightarrow$  and  $\neg$  are.
- Examples:

*TomMarvoloRiddle* = *LordVoldemort*

*MorningStar* = *EveningStar*

- Equality can only be applied to **objects**; to state that two **propositions** are equal, use  $\leftrightarrow$ .

Let's see some more examples.

$$\begin{aligned} & \text{FavoriteMovieOf}(You) \neq \text{FavoriteMovieOf}(Date) \wedge \\ & \text{StarOf}(\text{FavoriteMovieOf}(You)) = \text{StarOf}(\text{FavoriteMovieOf}(Date)) \end{aligned}$$

*FavoriteMovieOf(You) ≠ FavoriteMovieOf(Date) ∧  
StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))*

$$\text{FavoriteMovieOf}(\text{You}) \neq \text{FavoriteMovieOf}(\text{Date}) \wedge \\ \text{StarOf}(\text{FavoriteMovieOf}(\text{You})) = \text{StarOf}(\text{FavoriteMovieOf}(\text{Date}))$$

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These purple terms are **functions**. Functions take objects as input and produce objects as output.

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*FavoriteMovieOf(You) ≠ FavoriteMovieOf(Date) ∧*  
*StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))*

# Functions

- First-order logic allows ***functions*** that return objects associated with other objects.
- Examples:

*ColorOf(Money)*

*MedianOf(x, y, z)*

$x + y$

- As with predicates, functions can take in any number of arguments, but always return a single value.
- Functions evaluate to ***objects***, not ***propositions***.

# Objects and Propositions

- When working in first-order logic, be careful to keep objects (actual things) and propositions (true or false) separate.
- You cannot apply connectives to objects:

*Venus → TheSun*

- You cannot apply functions to propositions:

*StarOf(IsRed(Sun) ∧ IsGreen(Mars))*

- Ever get confused? *Just ask!*

# The Type-Checking Table

	... operate on ...	... and produce
Connectives ( $\leftrightarrow$ , $\wedge$ , etc.) ...	propositions	a proposition
Predicates ( $=$ , etc.) ...	objects	a proposition
Functions ...	objects	an object

One last (and major) change

Some bear is curious.

Some bear is curious.

$\exists b. (Bear(b) \wedge Curious(b))$

Some bear is curious.

$$\exists b. (Bear(b) \wedge Curious(b))$$

$\exists$  is the **existential quantifier**  
and says “there is a choice of  
b where the following is  
true.”

# The Existential Quantifier

- A statement of the form

$\exists x. \text{some-formula}$

is true when there exists a choice object where **some-formula** is true when that object is plugged in for x.

- Examples:

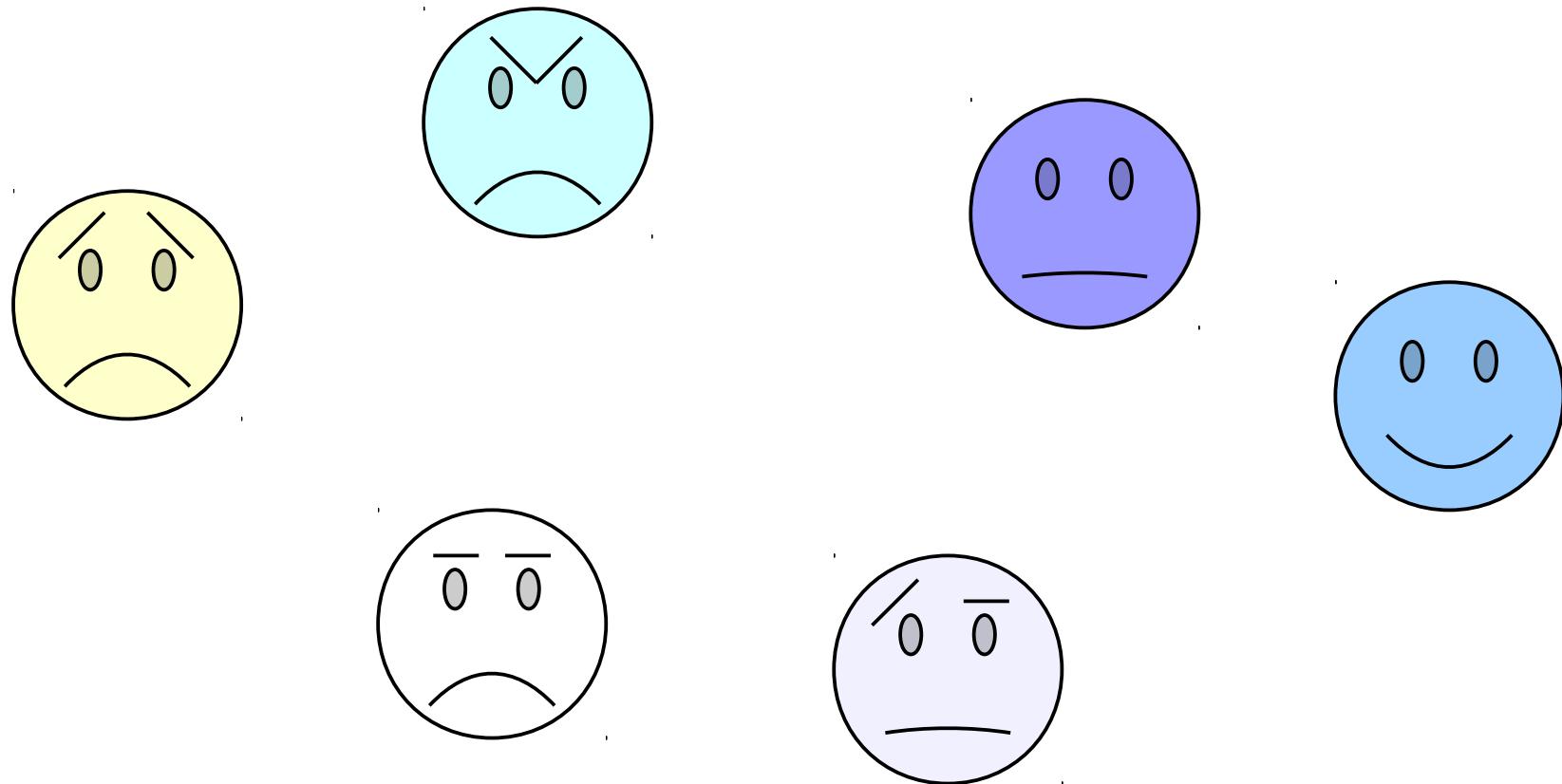
$\exists x. (Even(x) \wedge Prime(x))$

$\exists x. (TallerThan(x, me) \wedge WeighsLessThan(x, me))$

$(\exists w. Will(w)) \rightarrow (\exists x. Way(x))$

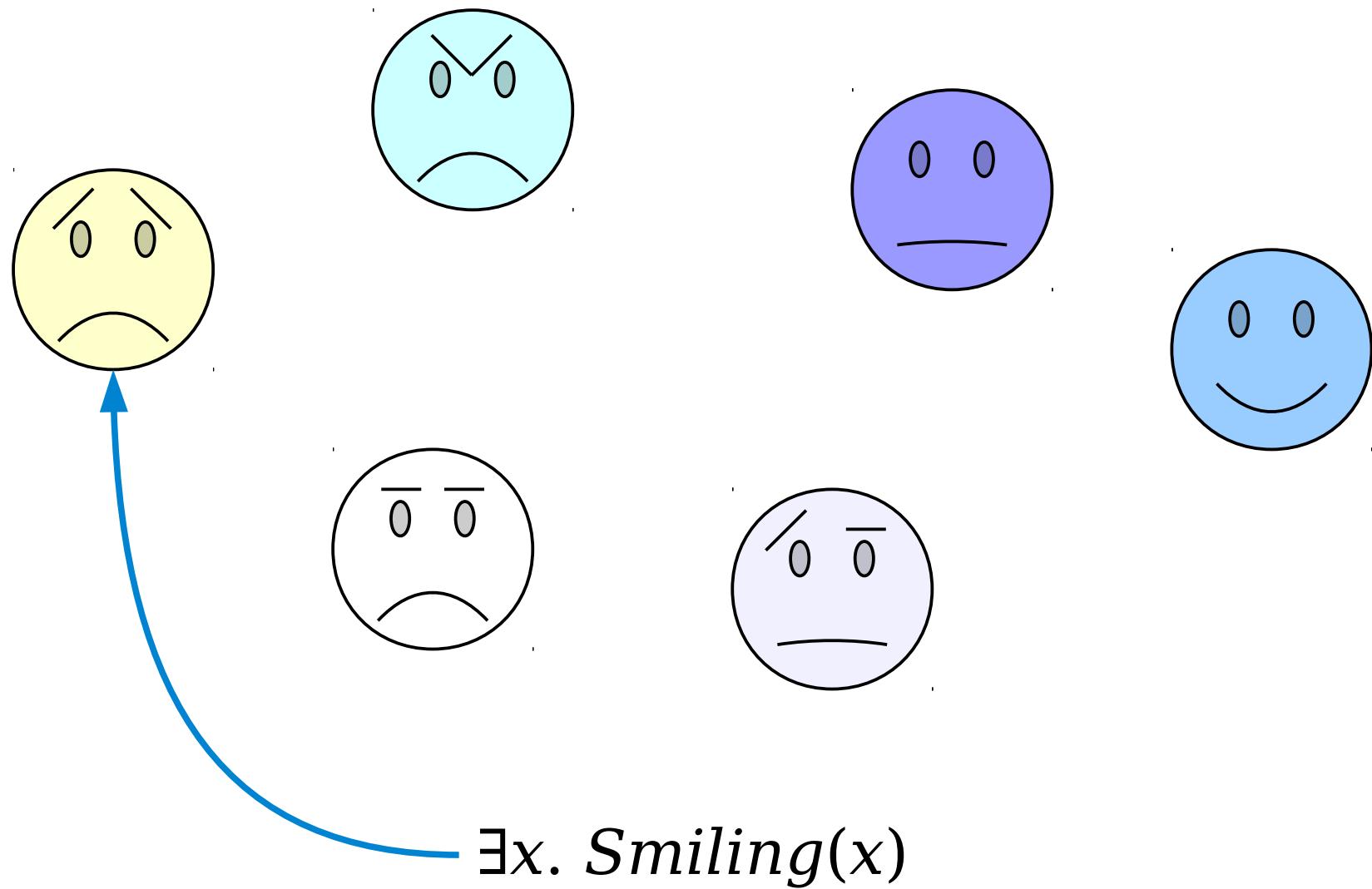
- Note the two ways of applying the  $\exists!$

# The Existential Quantifier

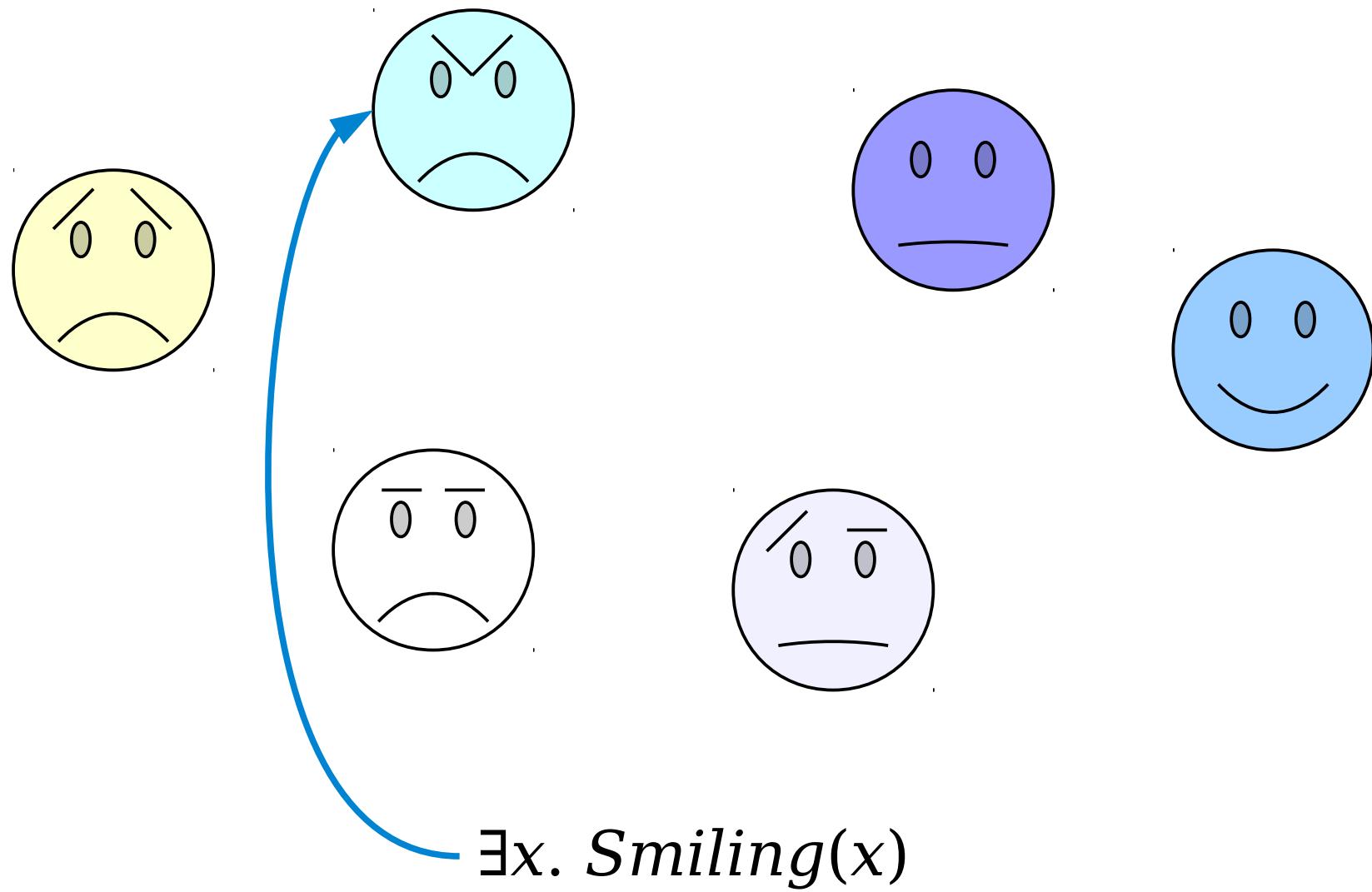


$\exists x. Smiling(x)$

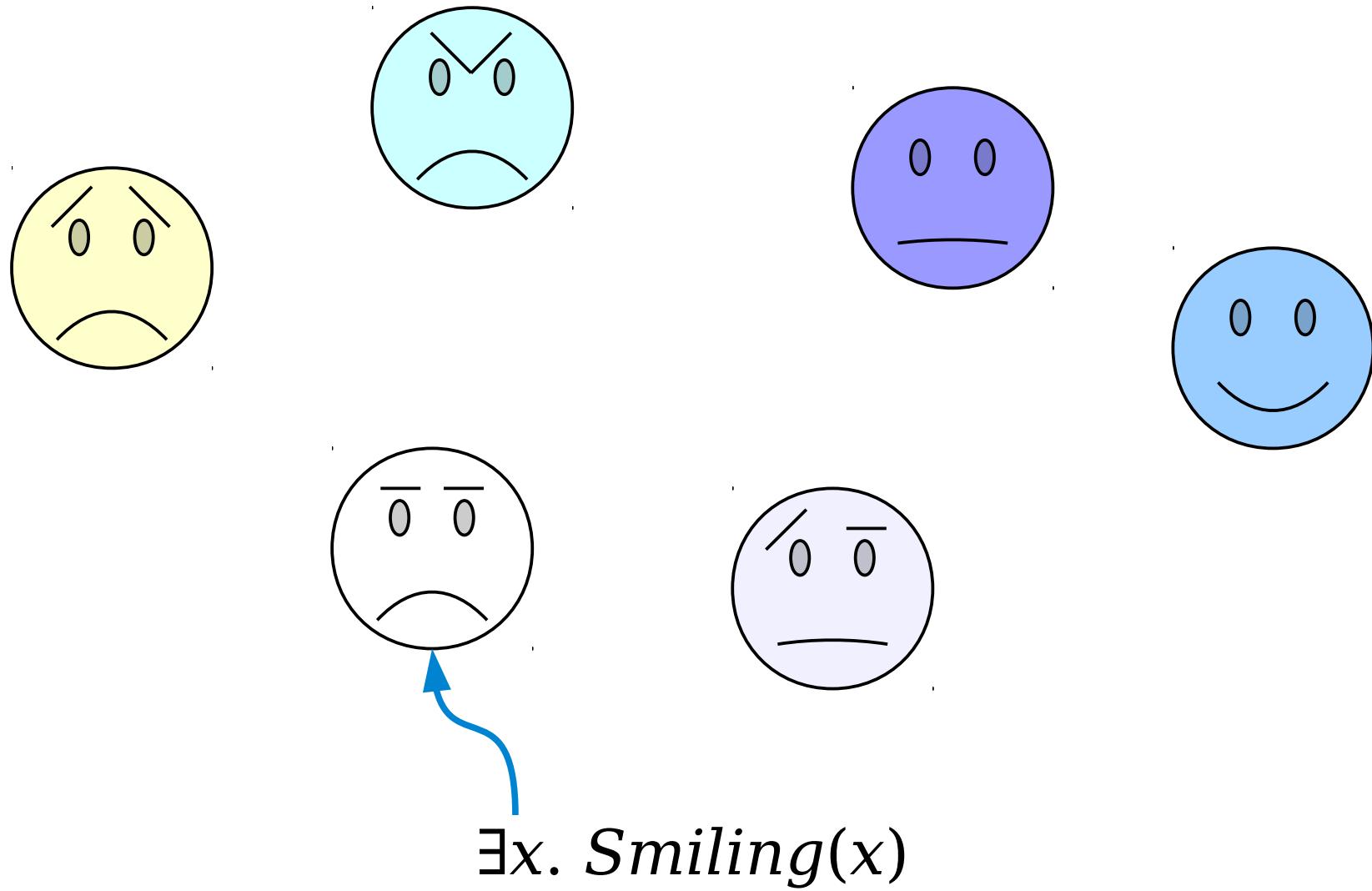
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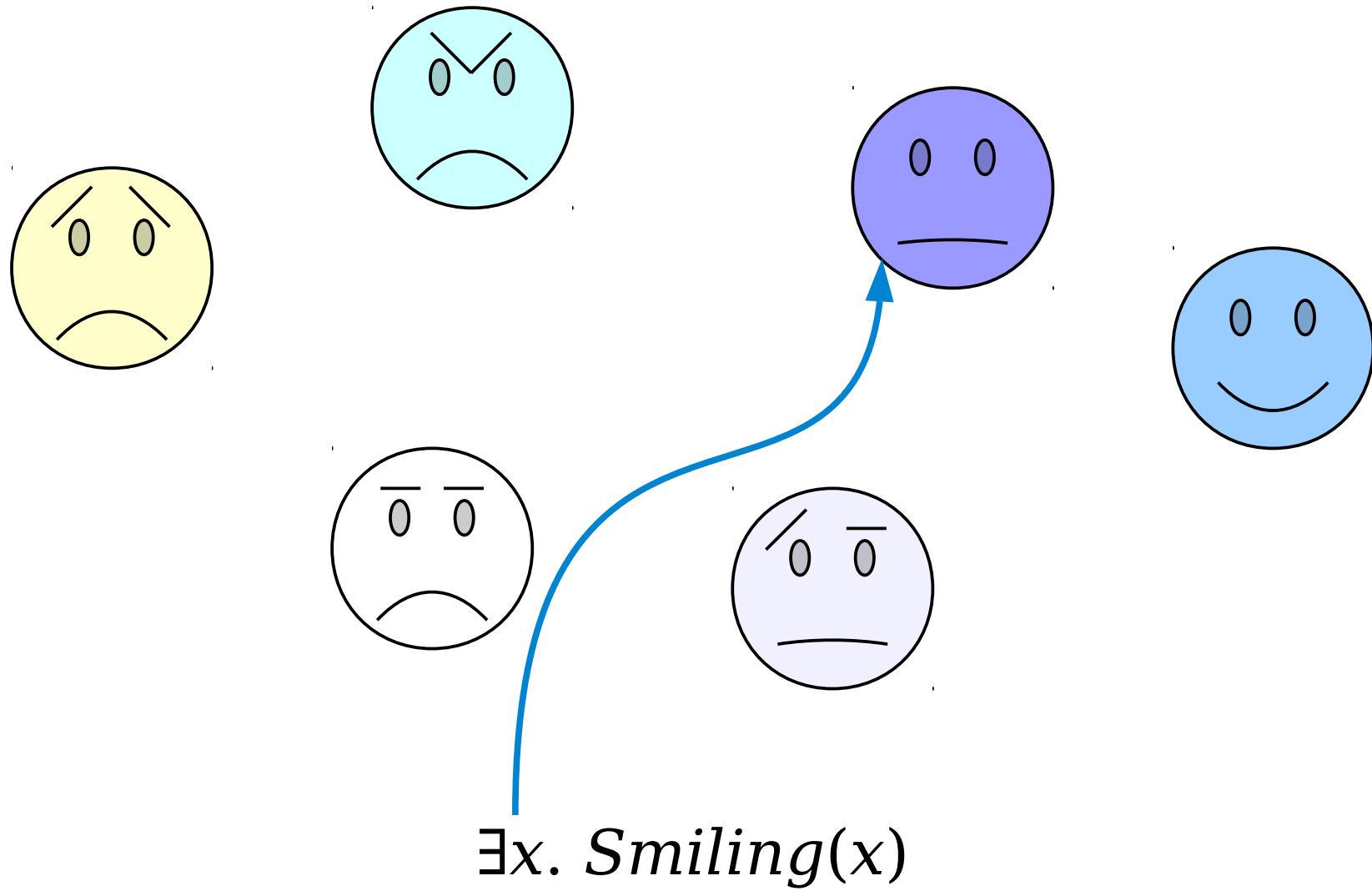
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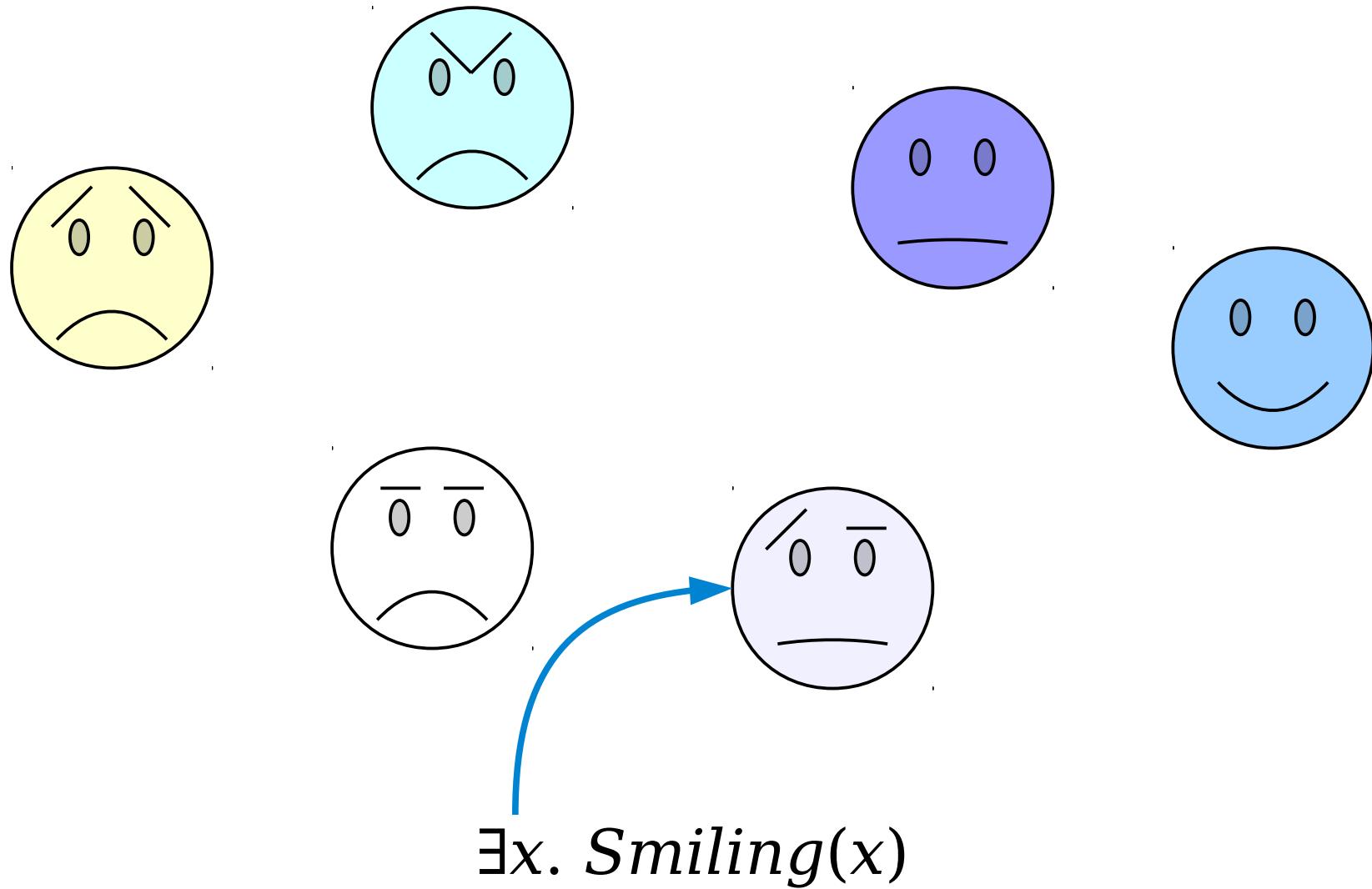
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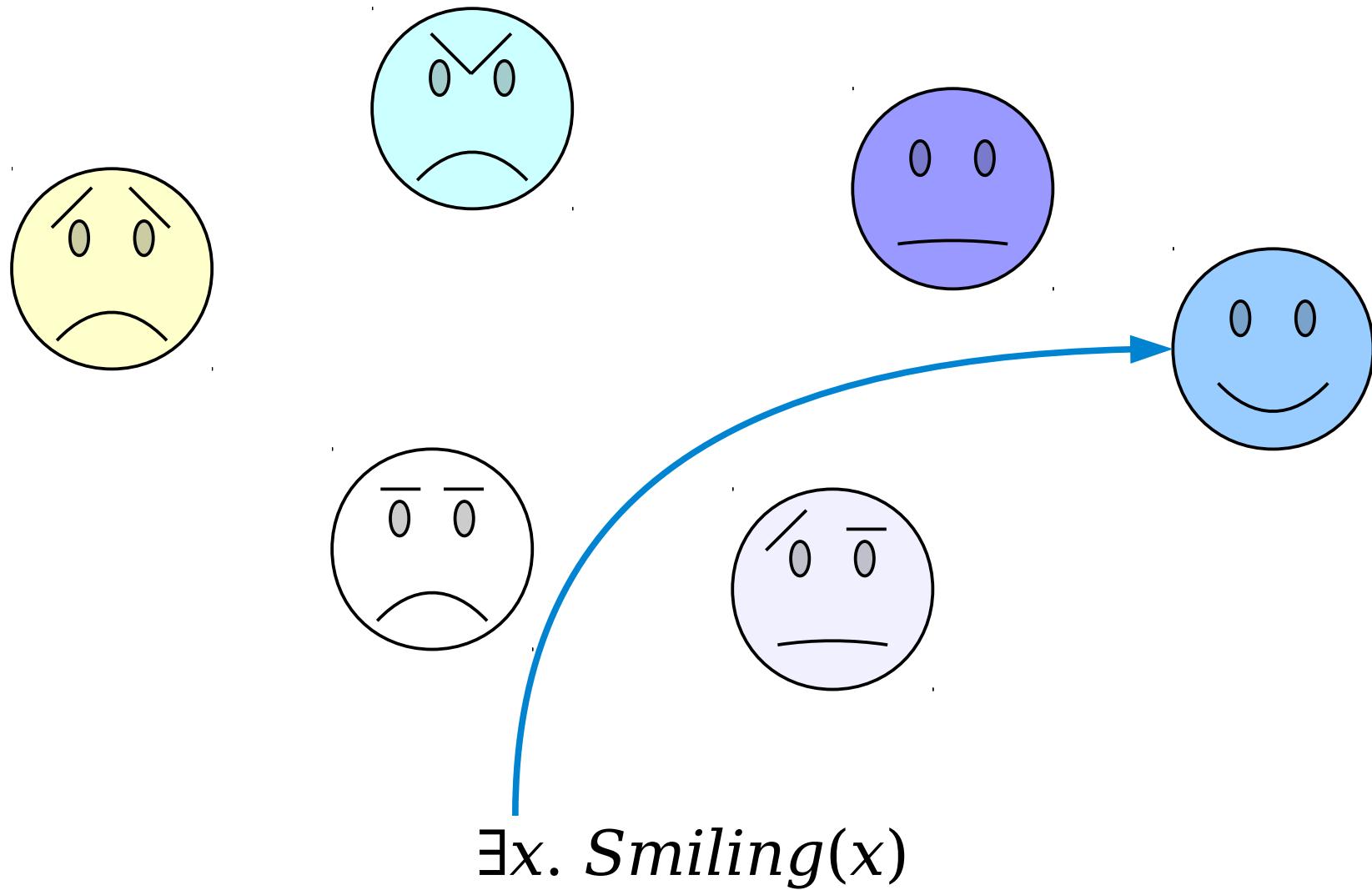
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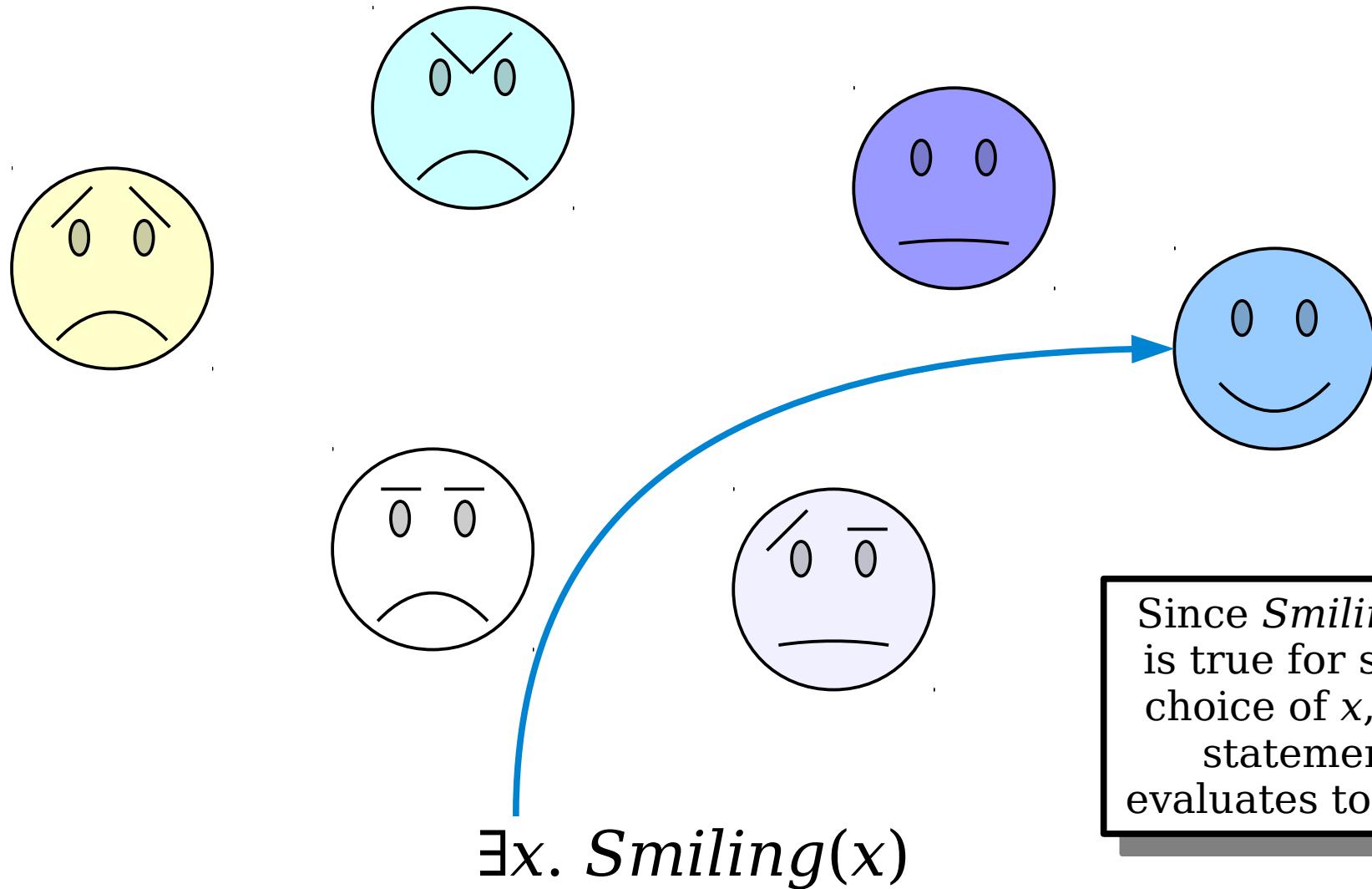
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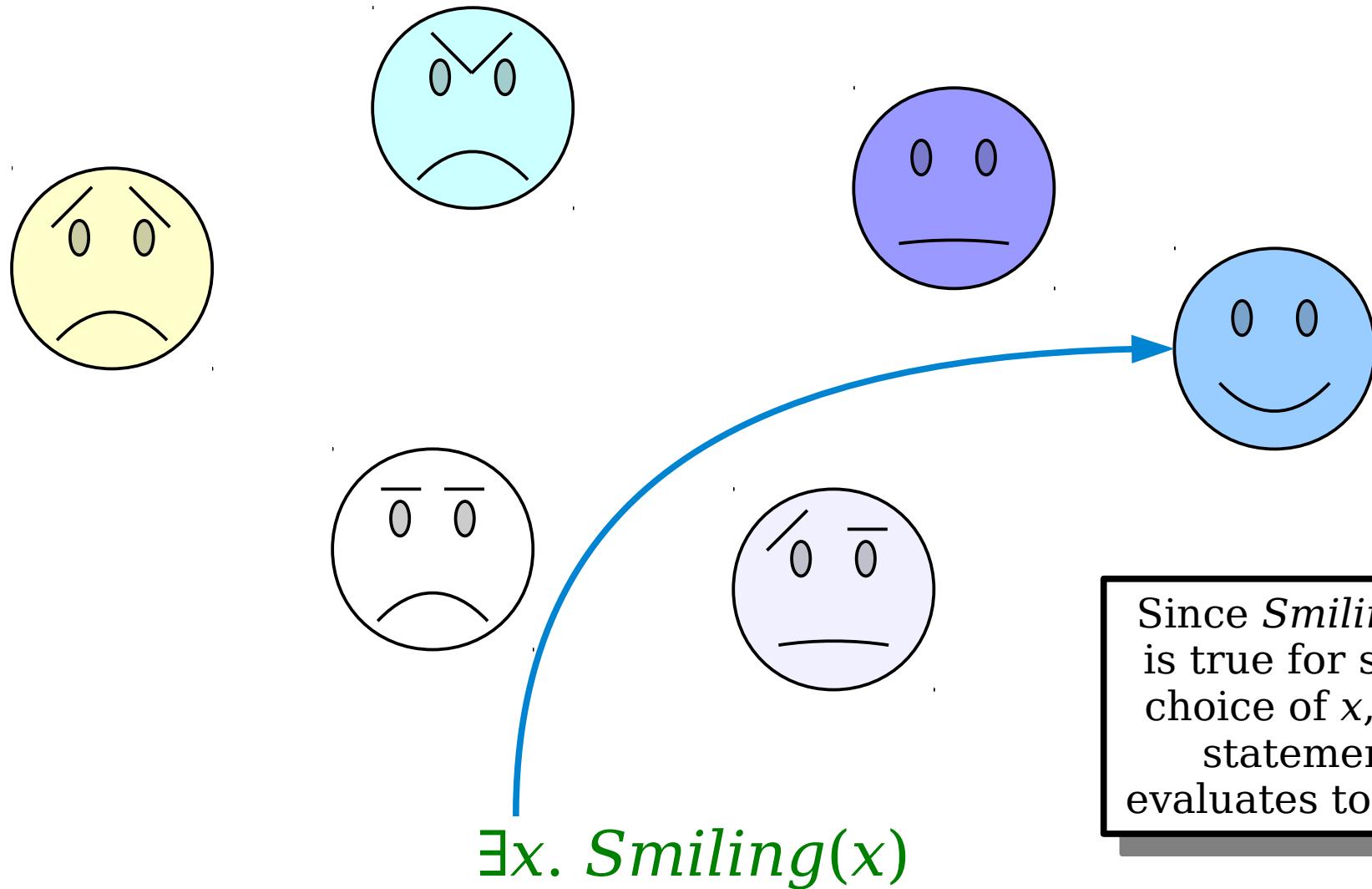
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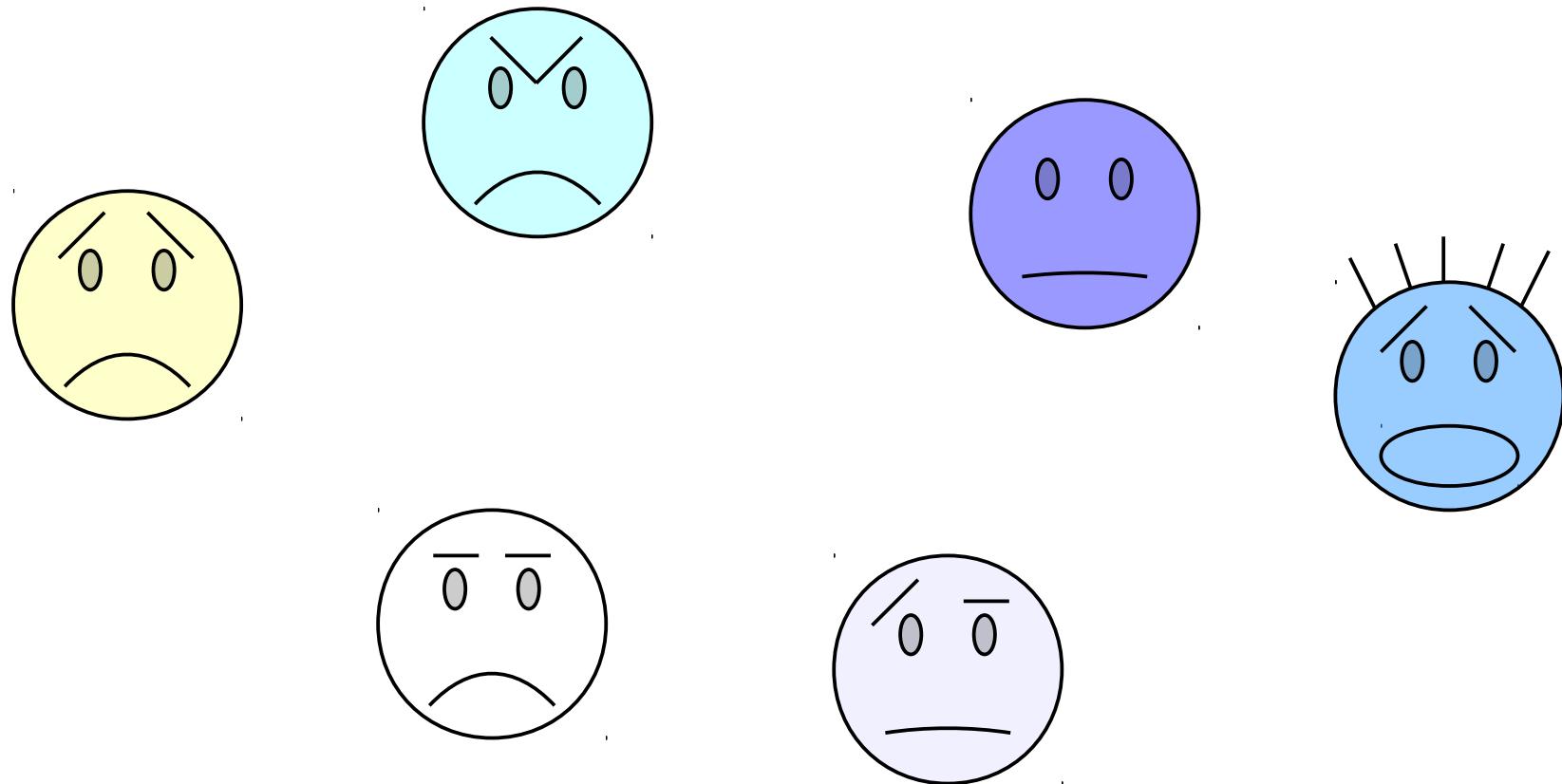
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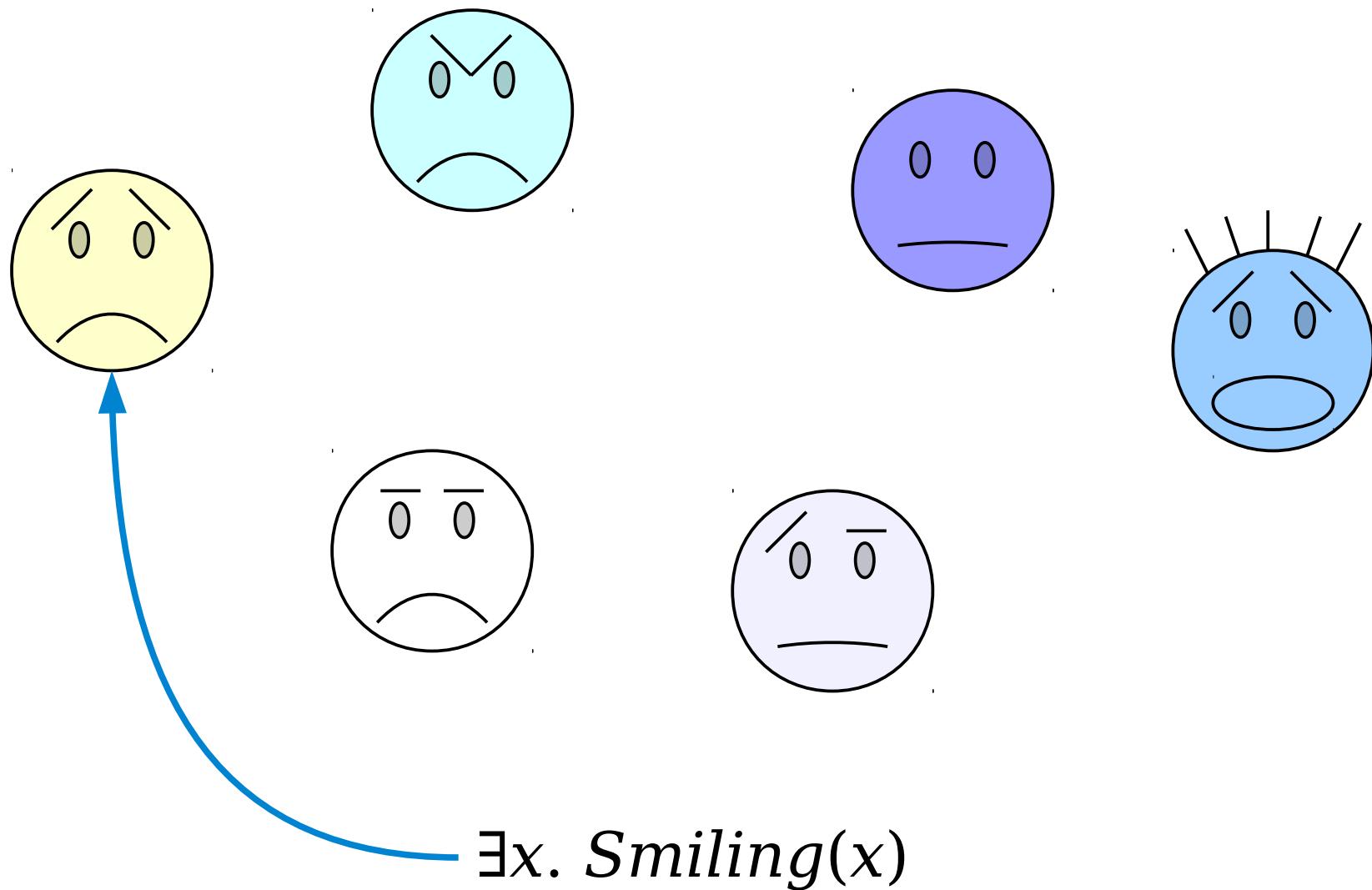


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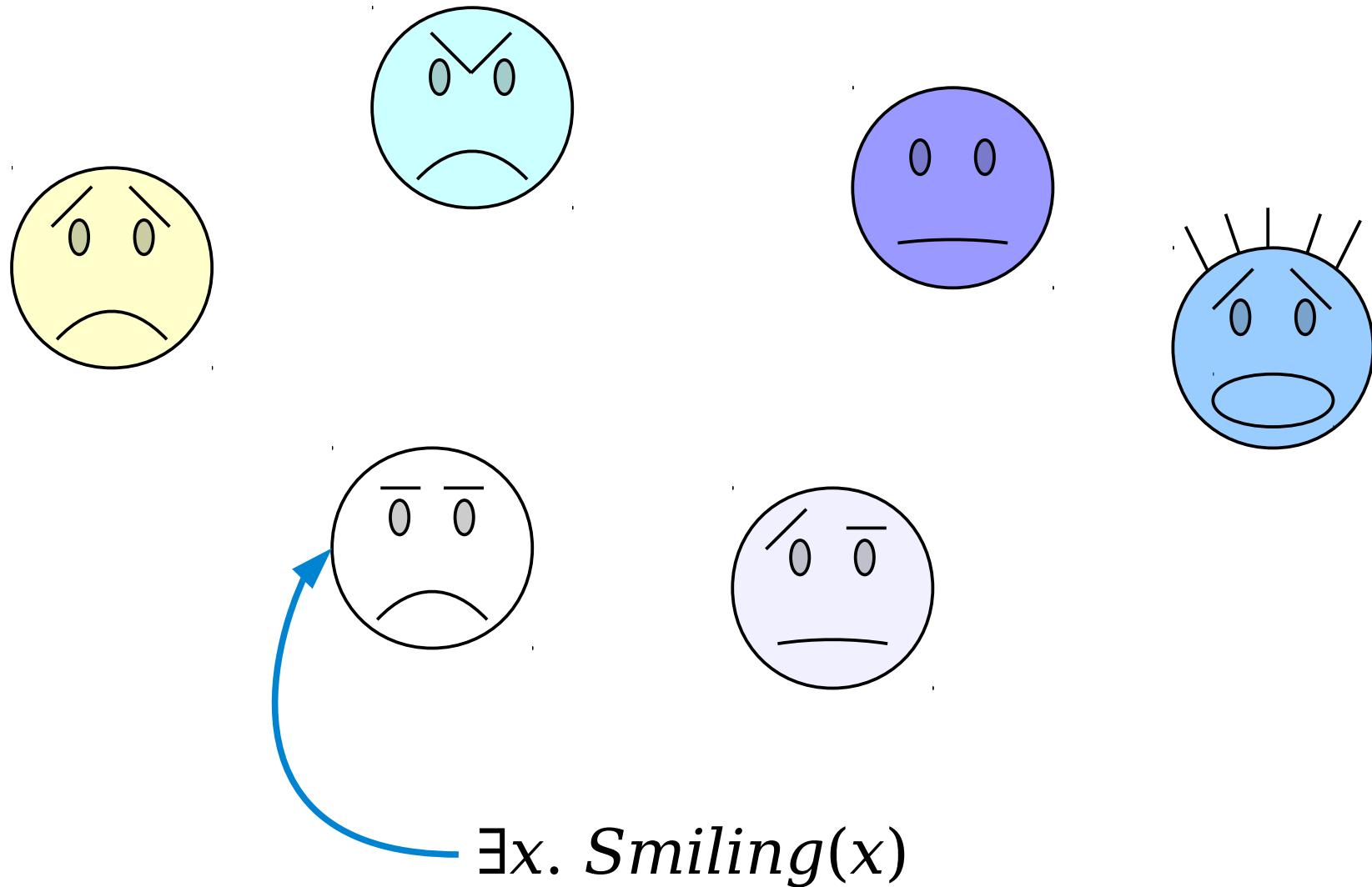


$\exists x. Smiling(x)$

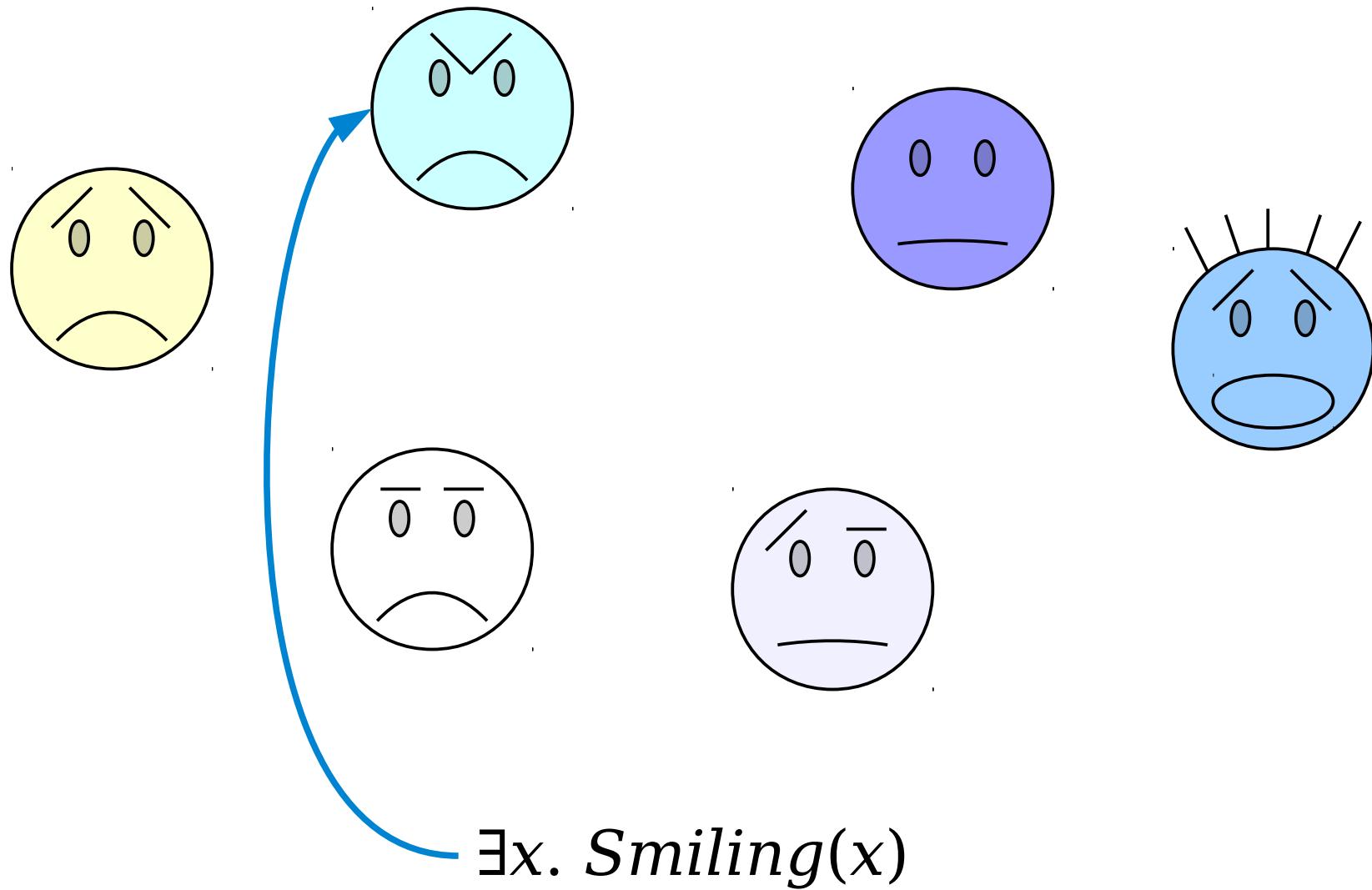
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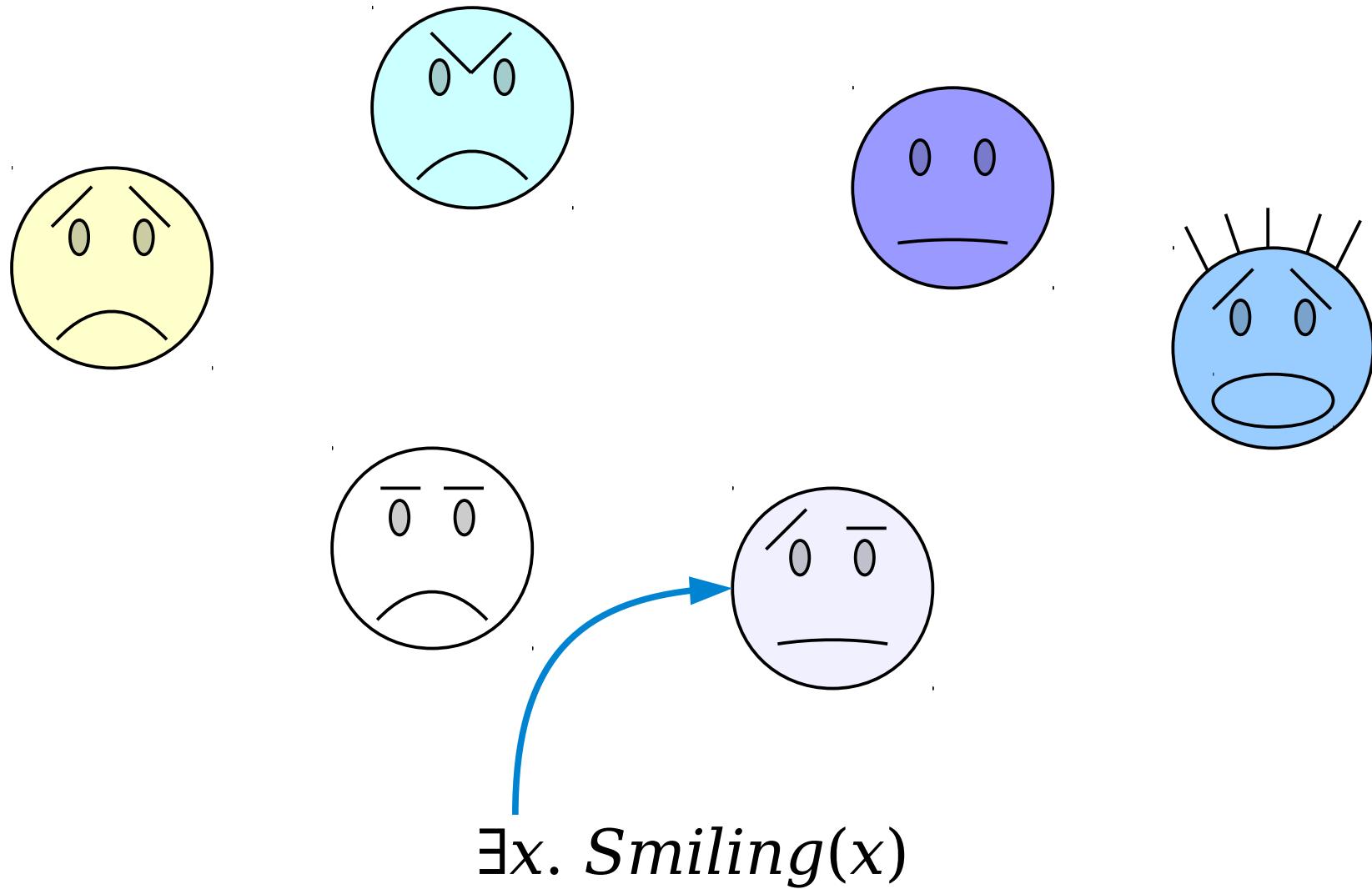
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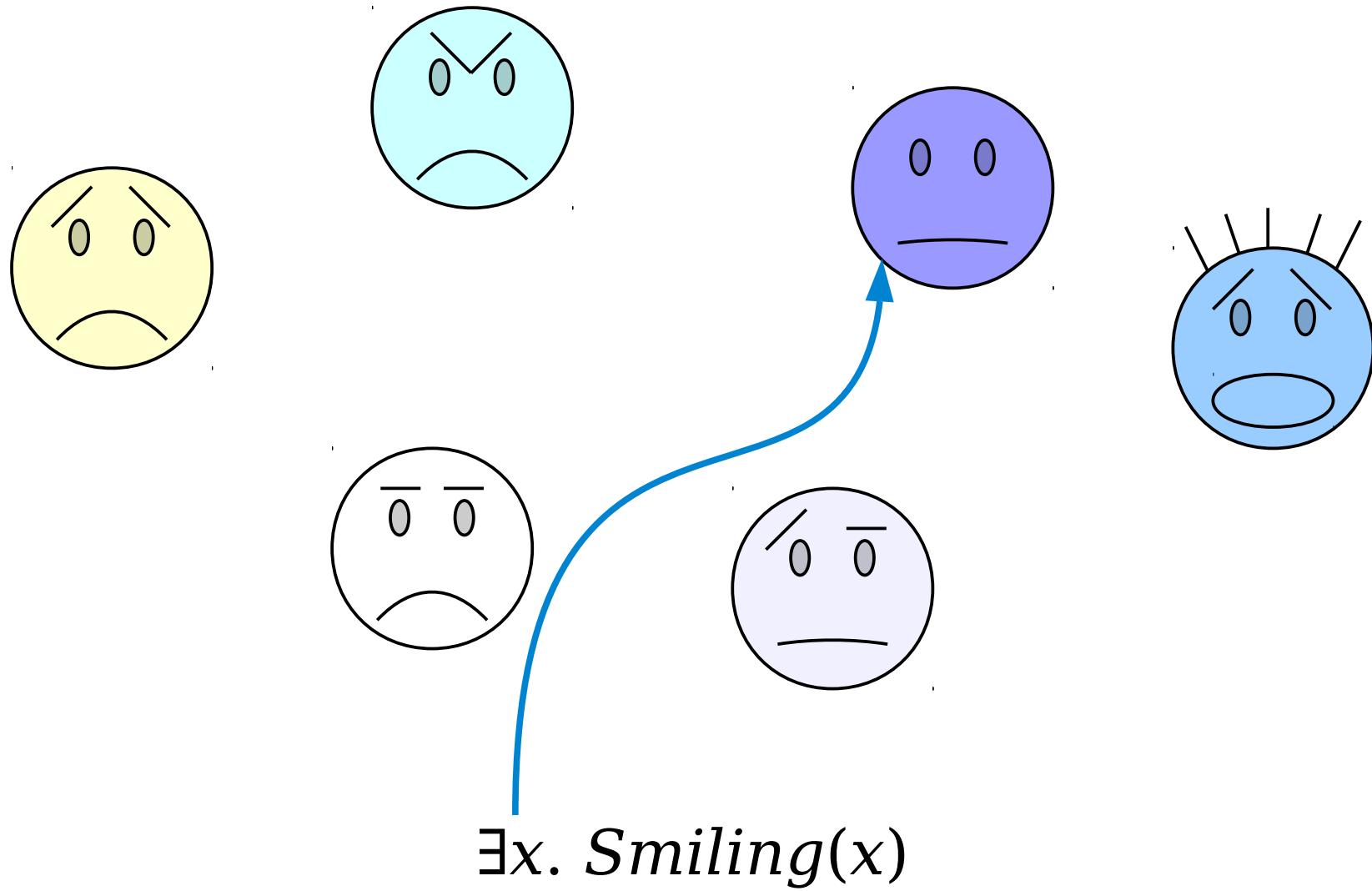
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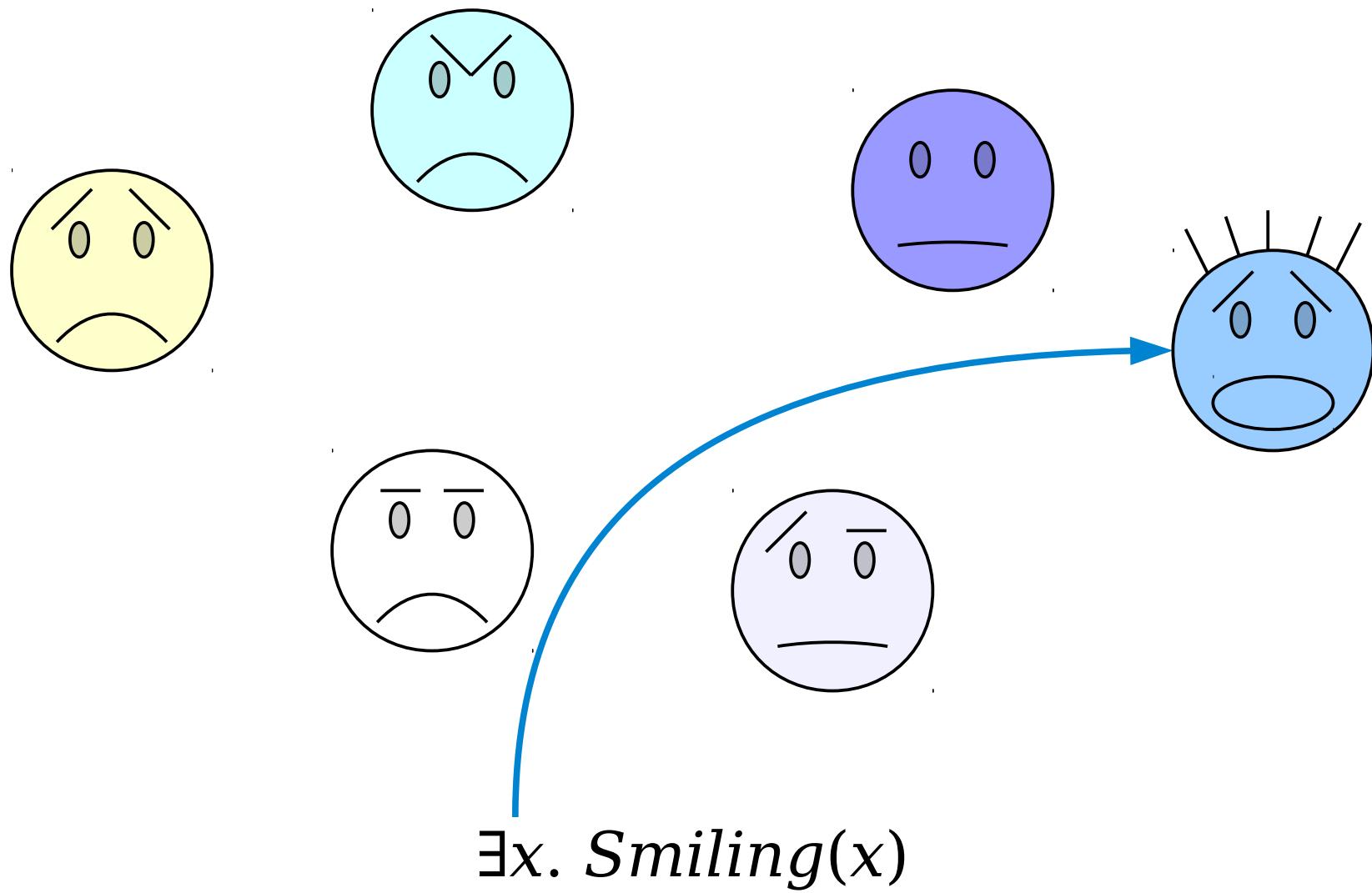
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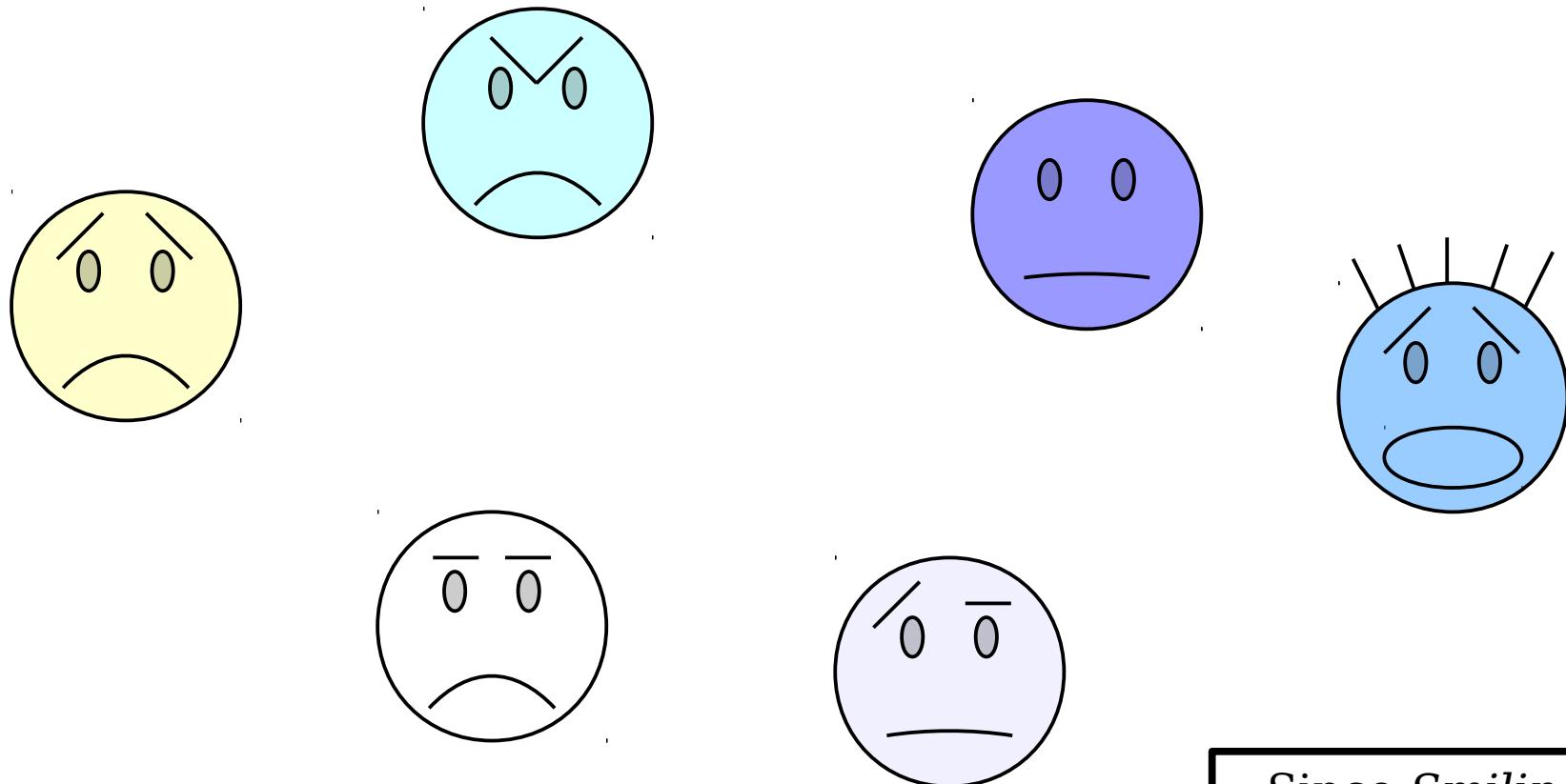
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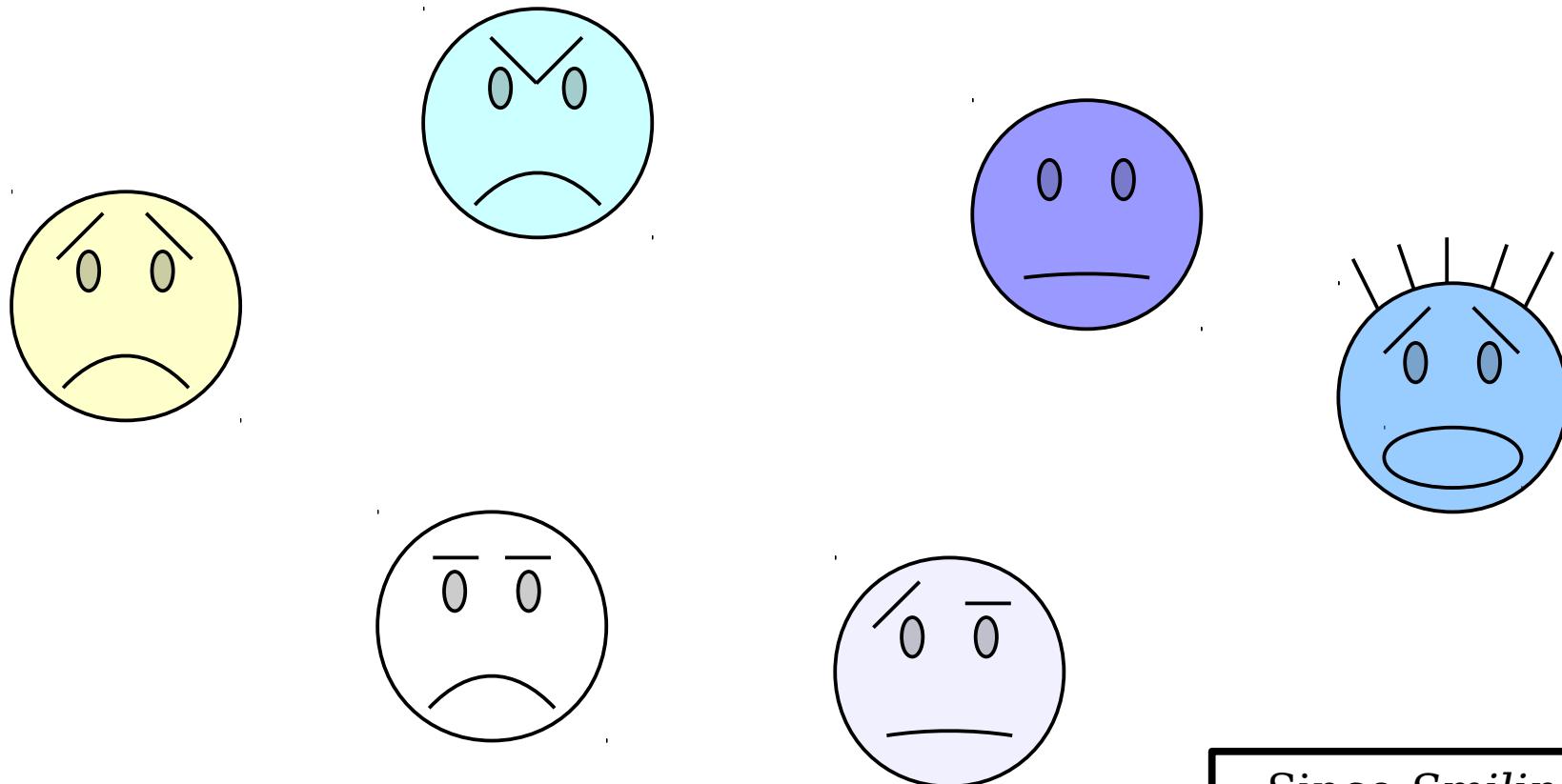
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$\exists x. Smiling(x)$

Since  $Smiling(x)$  is not true for any choice of  $x$ , this statement evaluates to false.

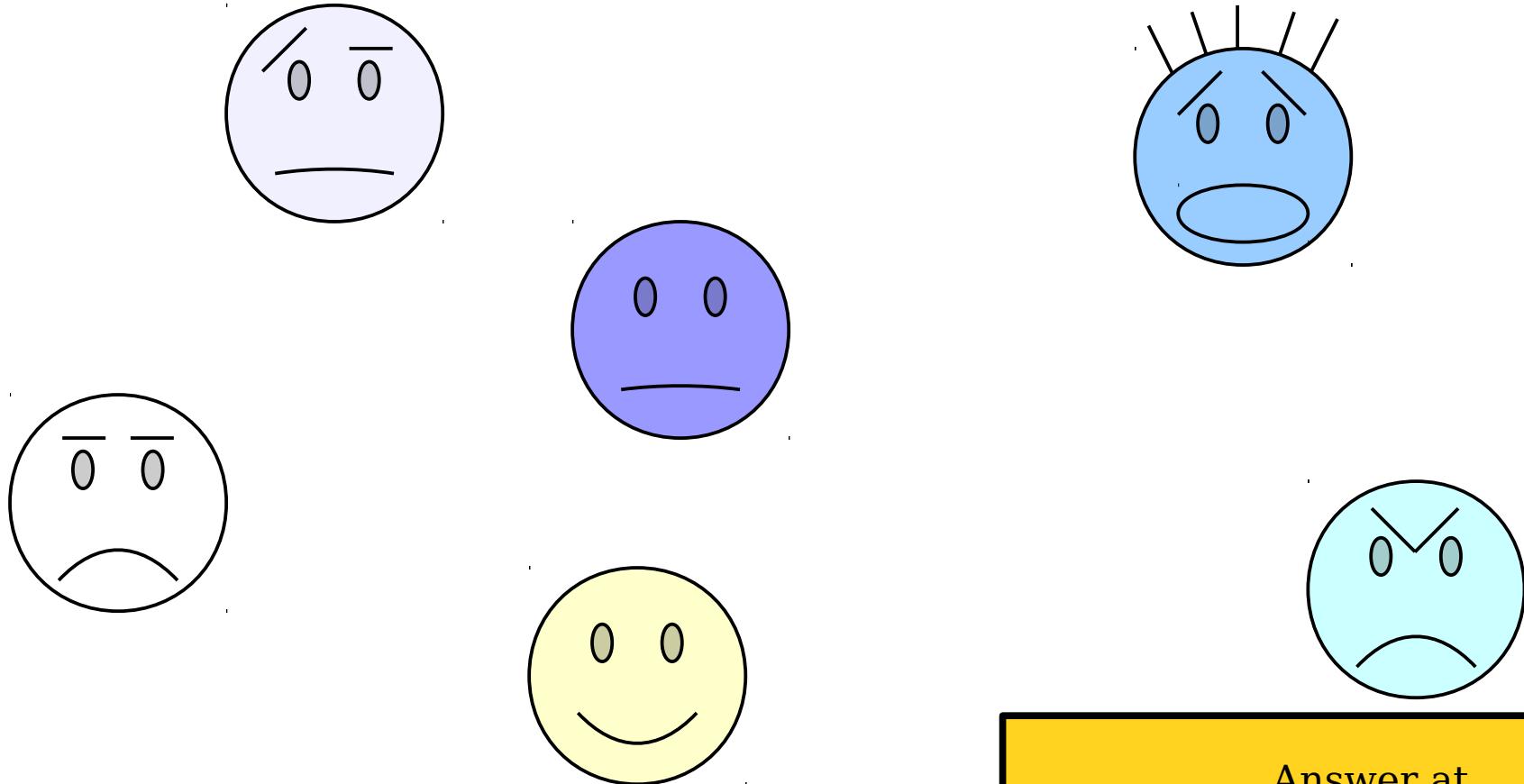
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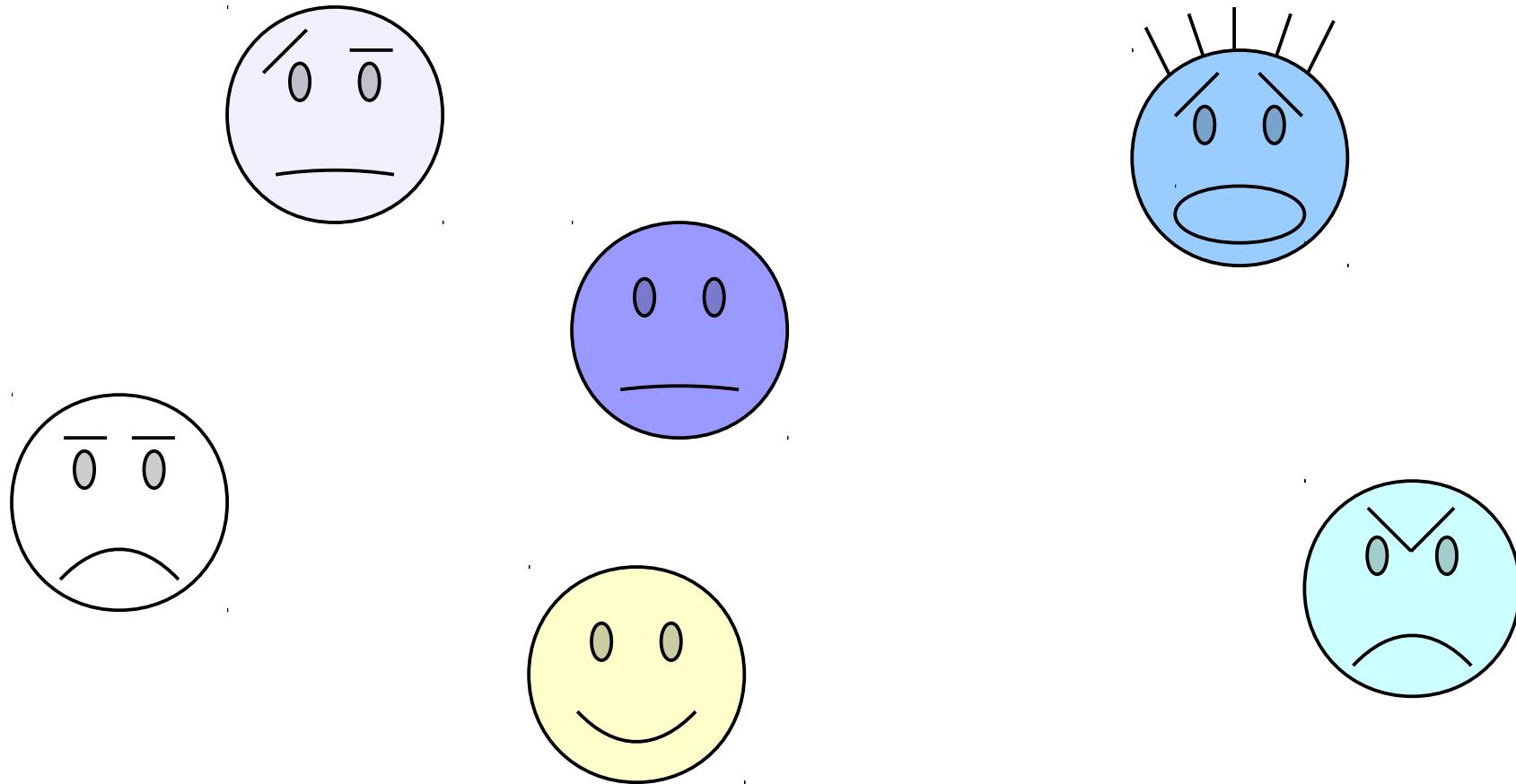
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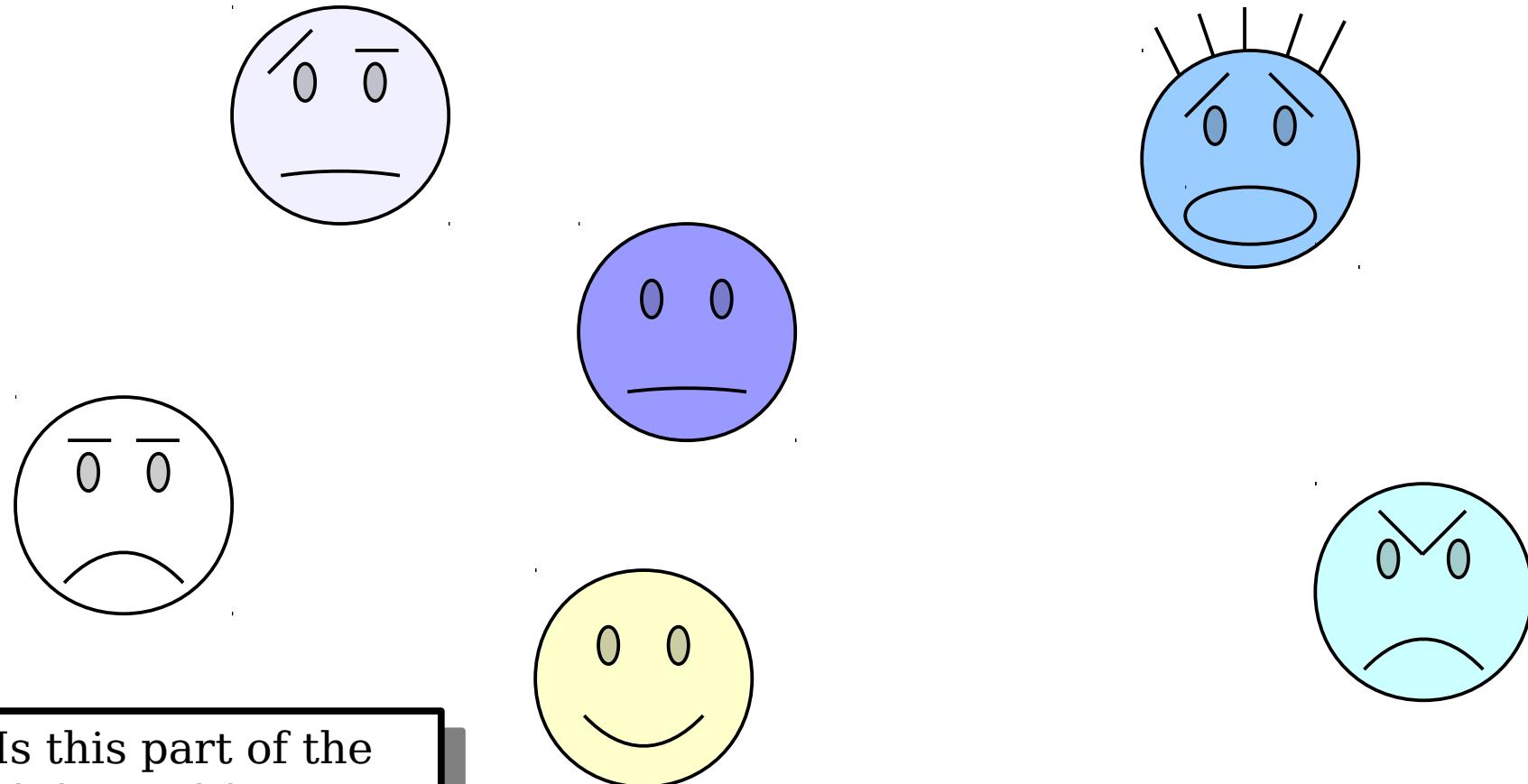
Answer at  
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$$(\exists x. Smiling(x)) \rightarrow (\exists y. WearingHat(y))$$

# The Existential Quantifier


$$(\exists x. Smiling(x)) \rightarrow (\exists y. WearingHat(y))$$

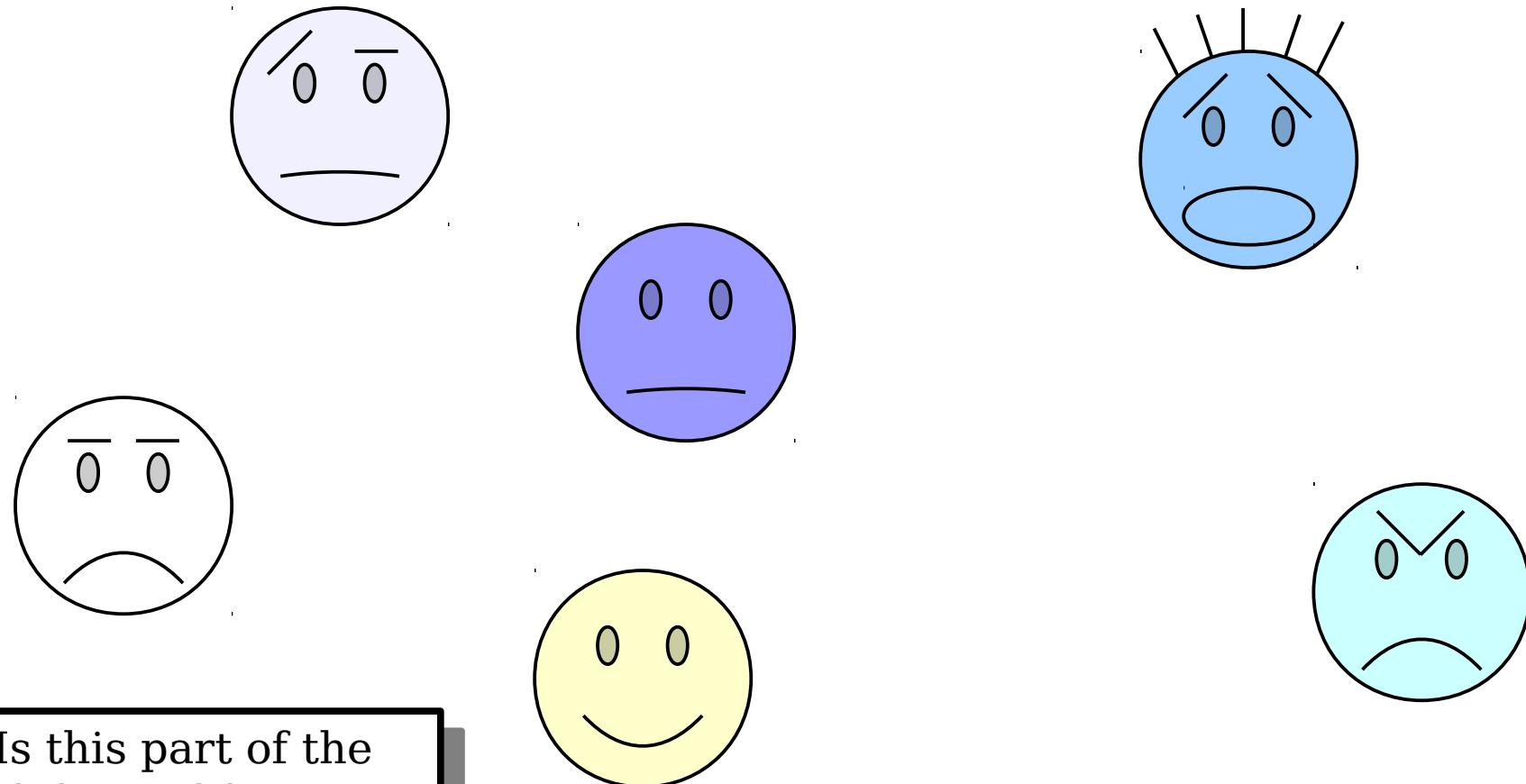
# The Existential Quantifier



Is this part of the statement true or false?

$$(\exists x. Smiling(x)) \rightarrow (\exists y. WearingHat(y))$$

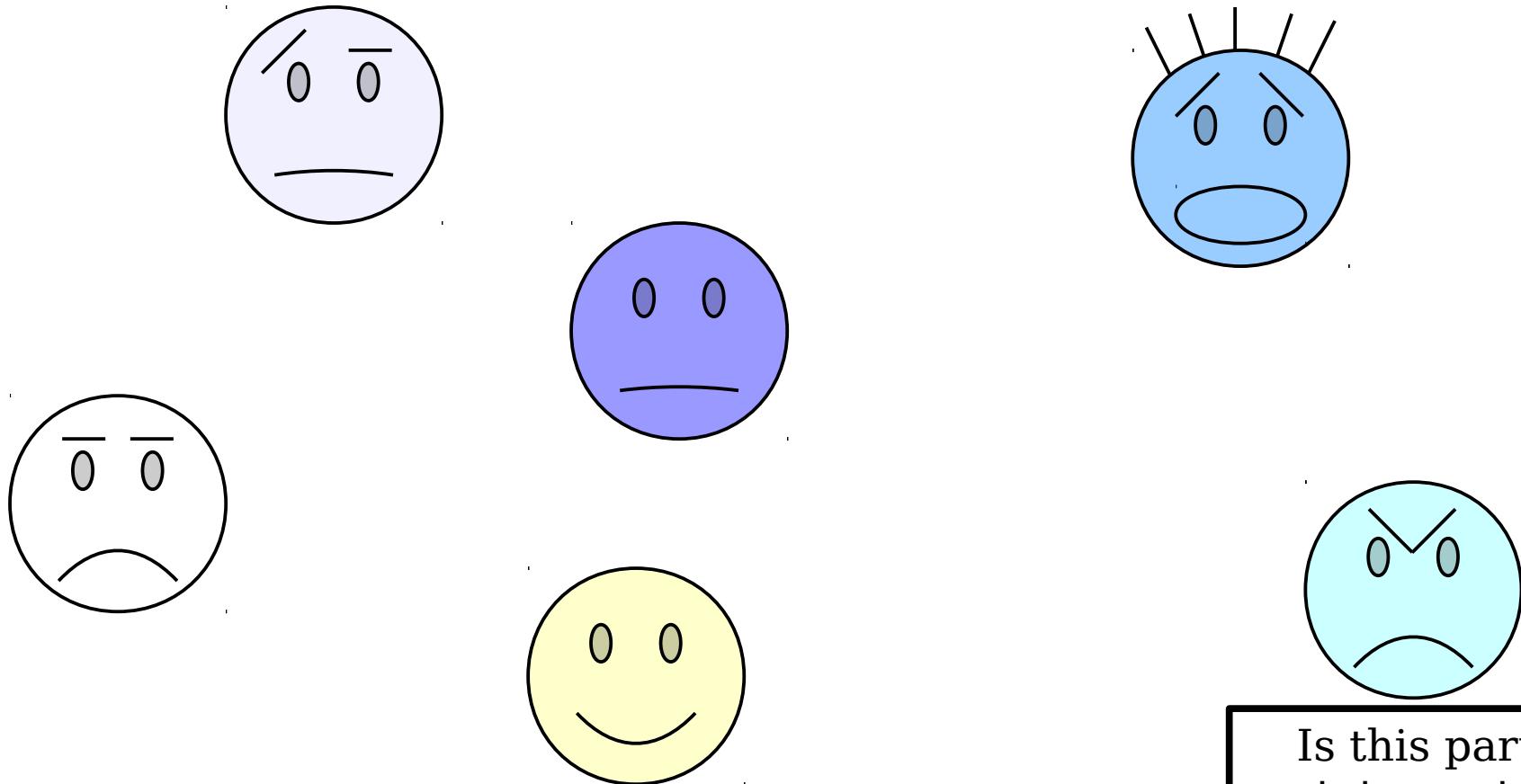
# The Existential Quantifier



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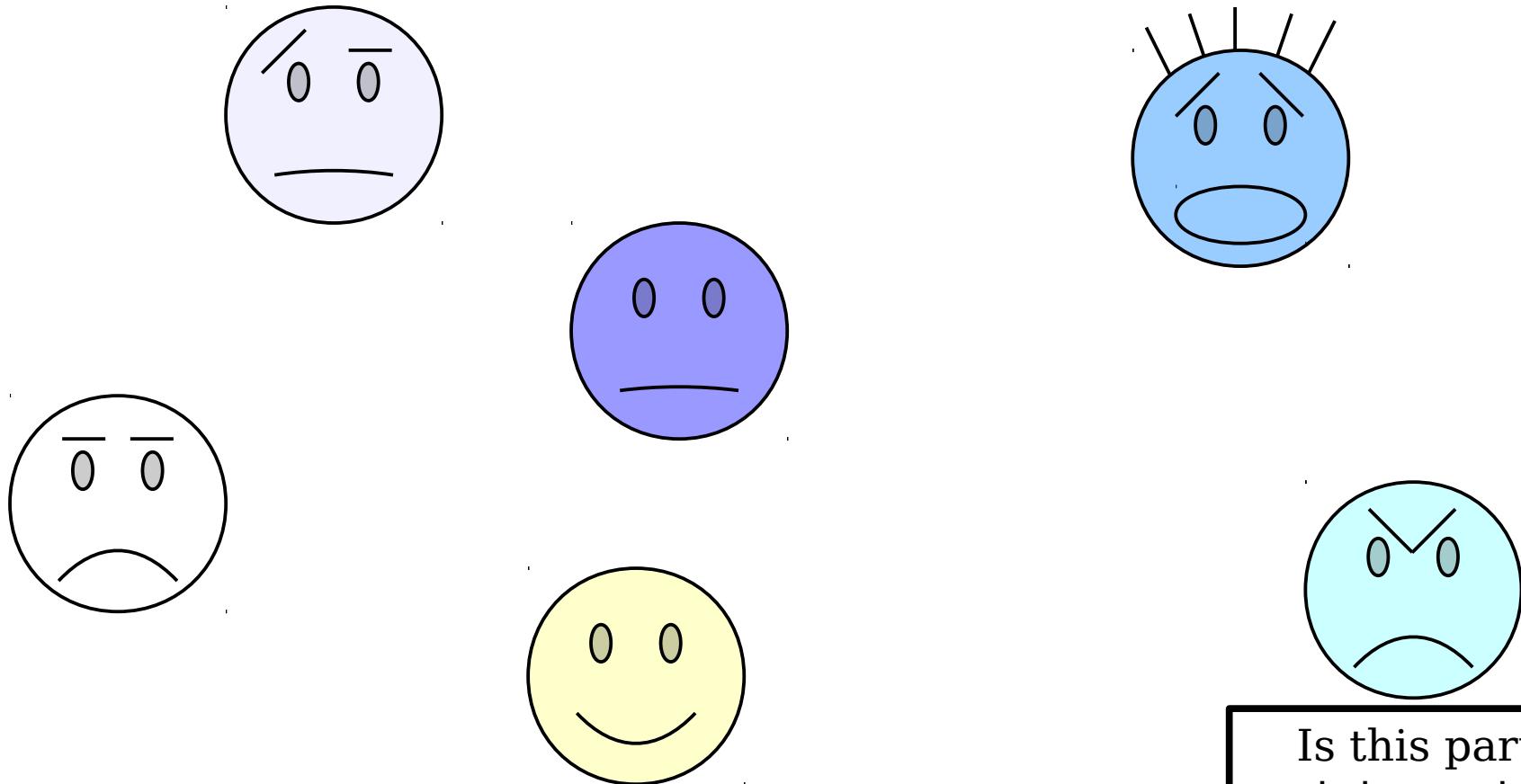
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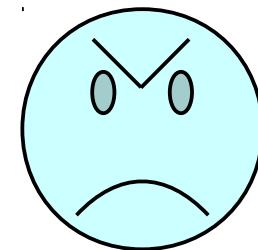
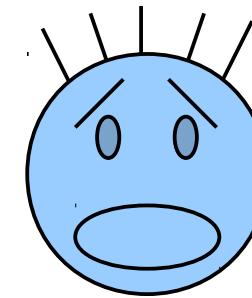
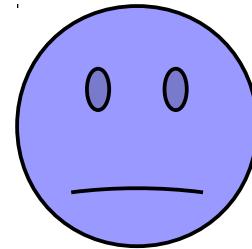
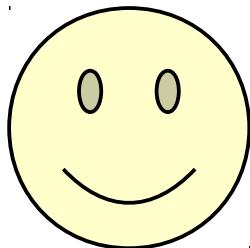
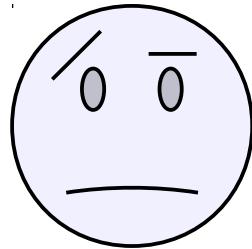
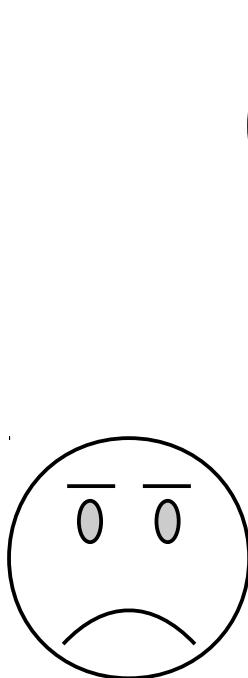
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$(\exists x. Smiling(x)) \rightarrow (\exists y. WearingHat(y))$

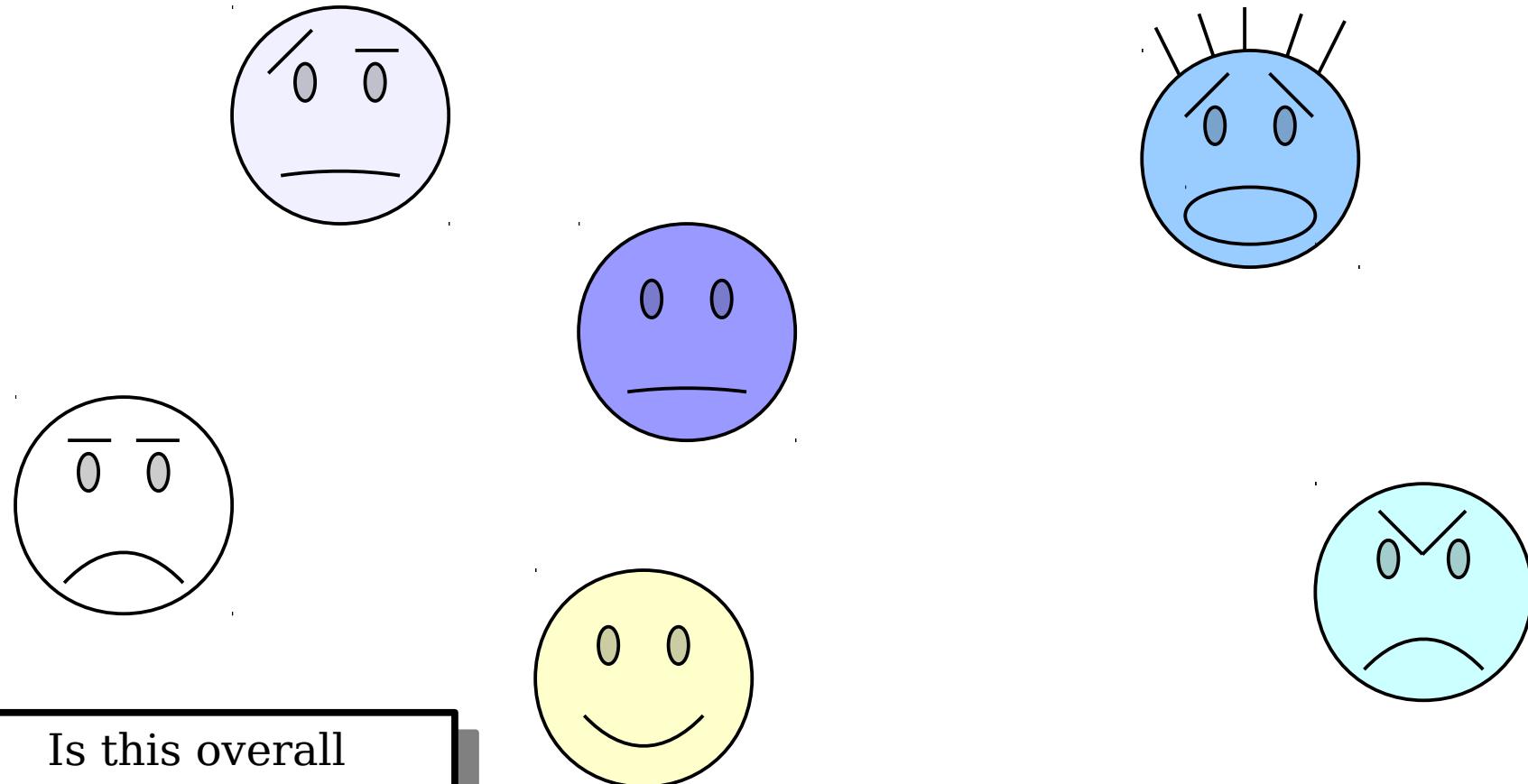
# The Existential Quantifier



Is this overall statement true or false?

$(\exists x. Smiling(x)) \rightarrow (\exists y. \cancel{WearingHat(y)})$

# The Existential Quantifier



Is this overall statement true or false?

~~$(\exists x. Smiling(x)) \rightarrow (\exists y. WearingHat(y))$~~

# Fun with Edge Cases

$\exists x. Smiling(x)$

# Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since nothing exists, period!

$\exists x. Smiling(x)$

# Some Technical Details

# Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.  
$$(\exists x. Loves(You, x)) \wedge (\exists y. Loves(y, You))$$

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The variable  $x$   
just lives here.

The variable  $y$   
just lives here.

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- The variable introduced is scoped just to the statement being quantified.

$$(\exists x. Loves(You, x)) \wedge (\exists x. Loves(x, You))$$

The variable  $x$   
just lives here.

A different variable,  
also named  $x$ , just  
lives here.

# Operator Precedence (Again)

- When writing out a formula in first-order logic, quantifiers have precedence just below  $\neg$ .
- The statement

$$\exists x. P(x) \wedge R(x) \wedge Q(x)$$

is parsed like this:

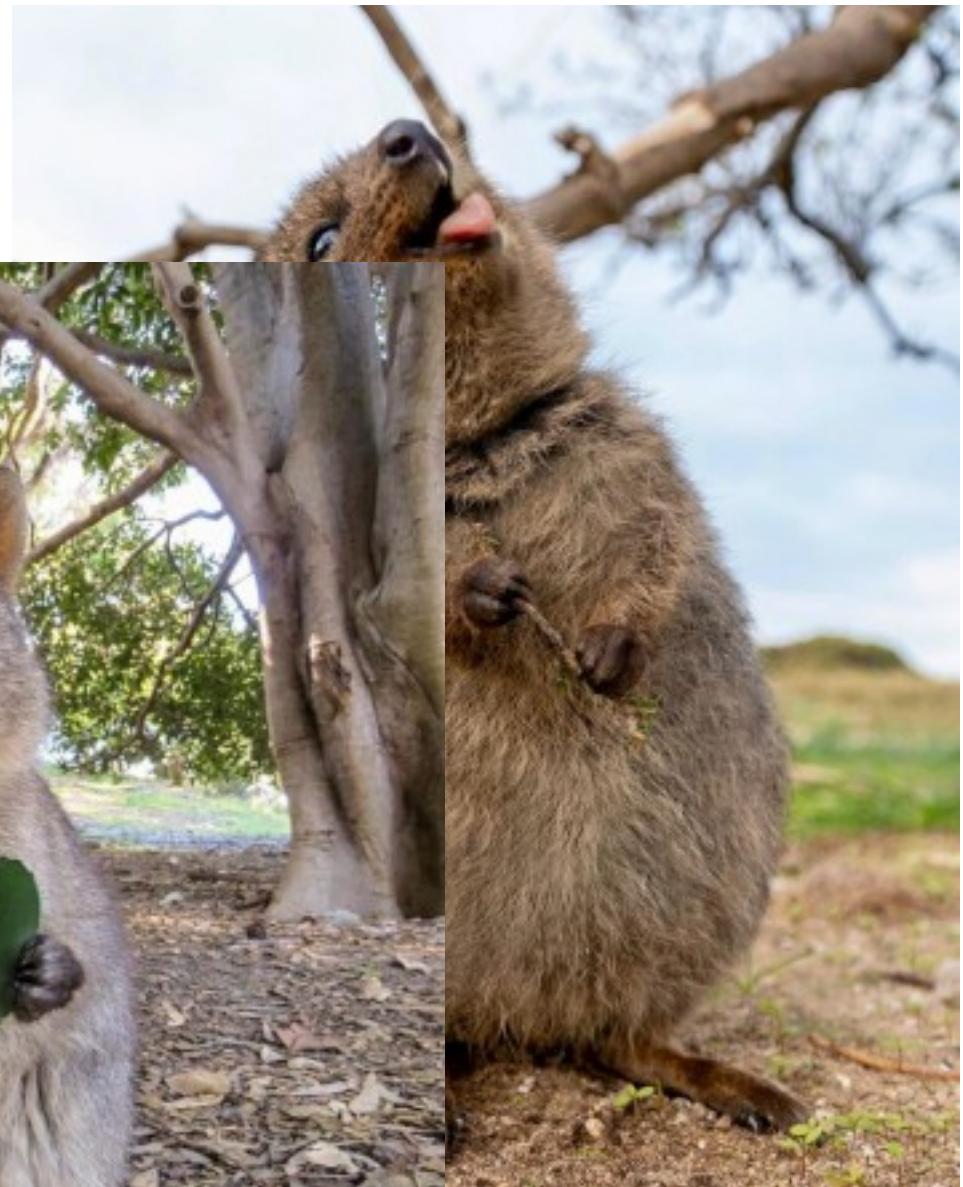
$$(\exists \textcolor{blue}{x}. P(\textcolor{blue}{x})) \wedge (R(\textcolor{red}{x}) \wedge Q(\textcolor{red}{x}))$$

- This is syntactically invalid because the variable  $x$  is out of scope in the back half of the formula.
- To ensure that  $x$  is properly quantified, explicitly put parentheses around the region you want to quantify:

$$\exists x. (P(x) \wedge R(x) \wedge Q(x))$$

**THIS IS A LOT!!**

Time out for cuteness overload.



Okay, back to CS103!

“For any natural number  $n$ ,  
 $n$  is even if and only if  $n^2$  is even”

“For any natural number  $n$ ,  
 $n$  is even if and only if  $n^2$  is even”

$$\forall n. (n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$$

“For any natural number  $n$ ,  
 $n$  is even if and only if  $n^2$  is even”

$\forall n. (n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$

$\forall$  is the ***universal quantifier***  
and says “for all choices of  $n$ ,  
the following is true.”

# The Universal Quantifier

- A statement of the form

**$\forall x. \text{some-formula}$**

is true when, for every choice of  $x$ , the statement ***some-formula*** is true when  $x$  is plugged into it.

- Examples:

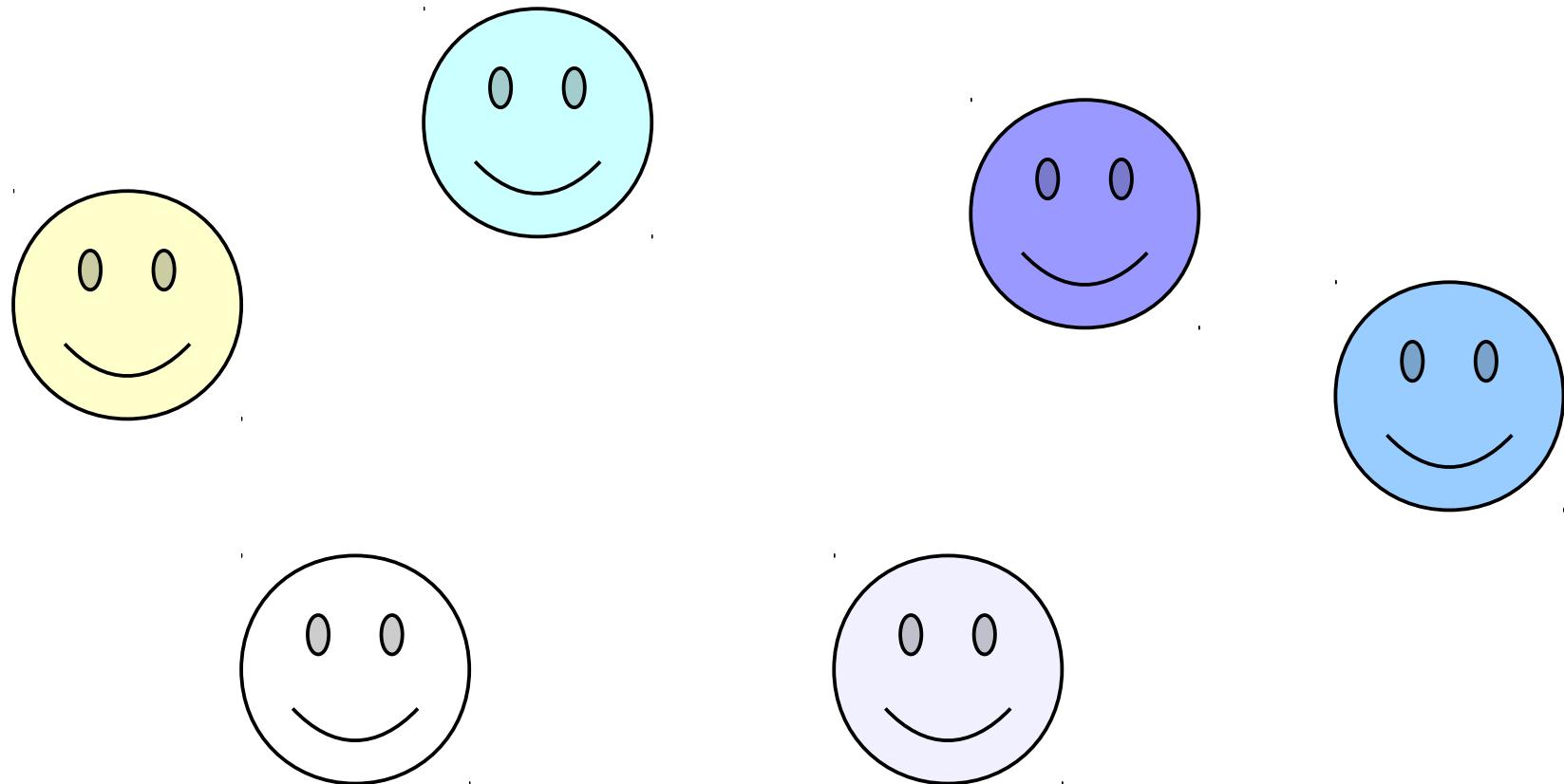
$\forall p. (\text{Puppy}(p) \rightarrow \text{Cute}(p))$

$\forall a. (\text{EatsPlants}(a) \vee \text{EatsAnimals}(a))$

$\text{Tallest}(\text{SultanKösen}) \rightarrow$

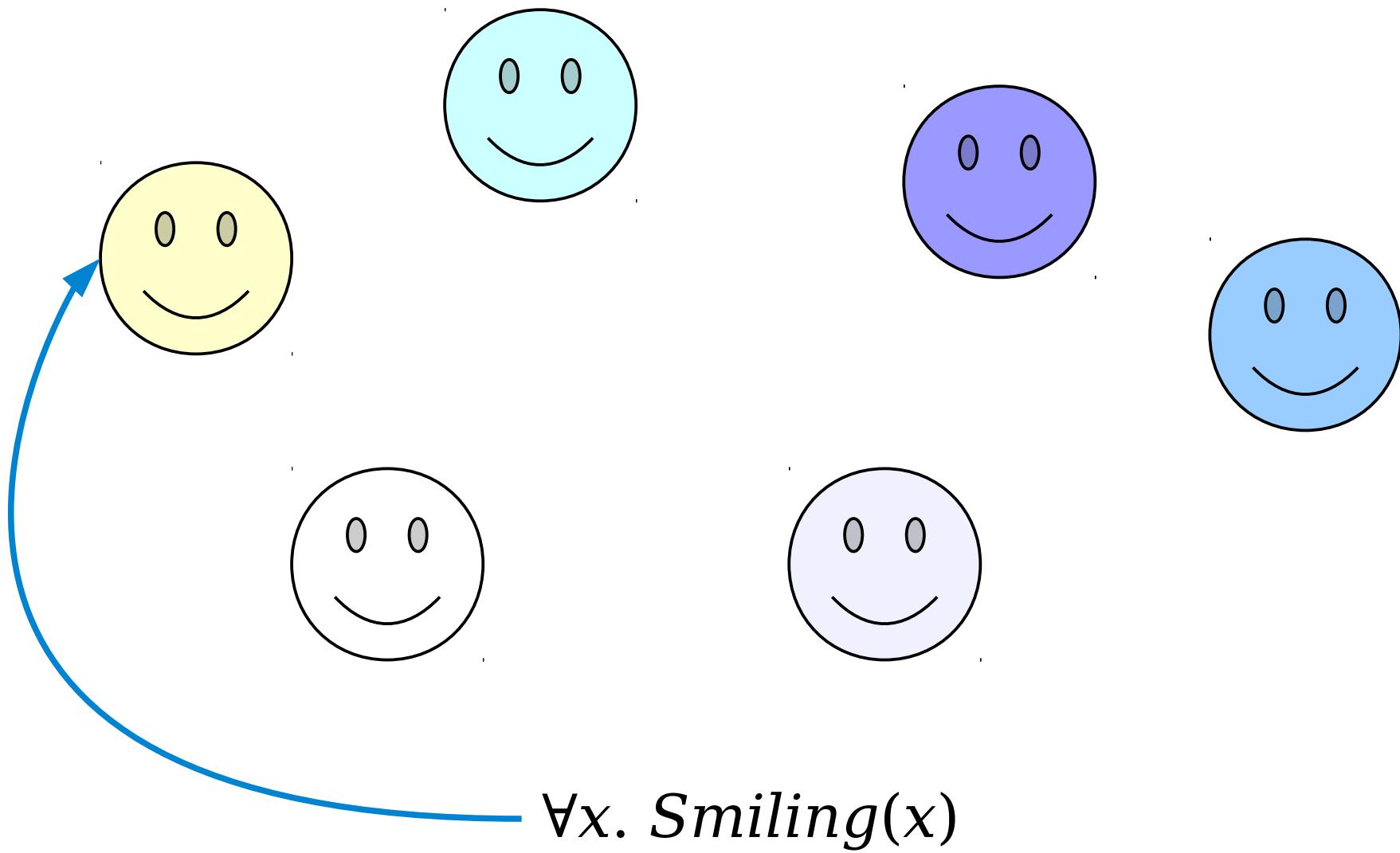
$\forall x. (\text{SultanKösen} \neq x \rightarrow \text{ShorterThan}(x, \text{SultanKösen}))$

# The Universal Quantifier

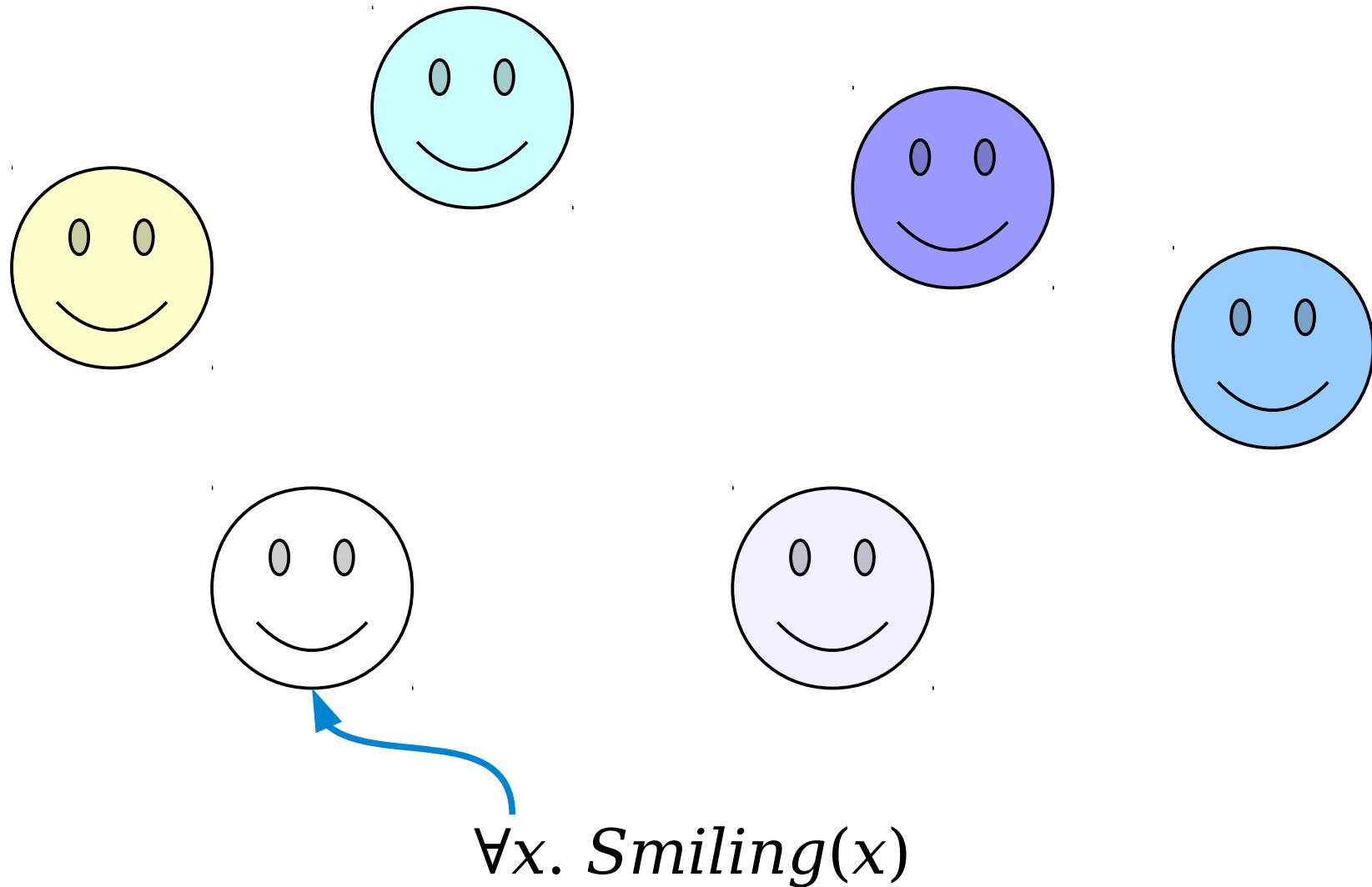


$\forall x. Smiling(x)$

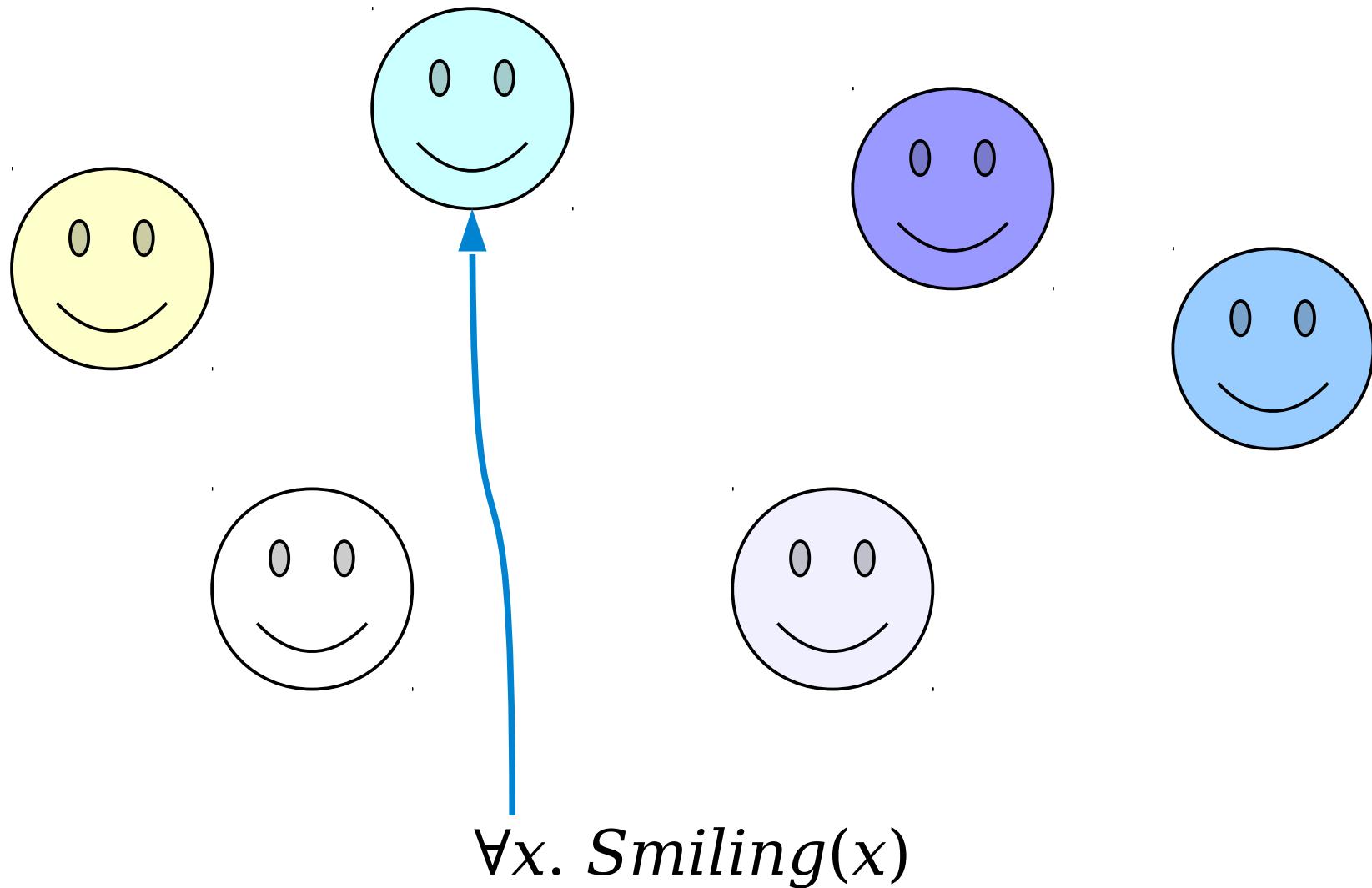
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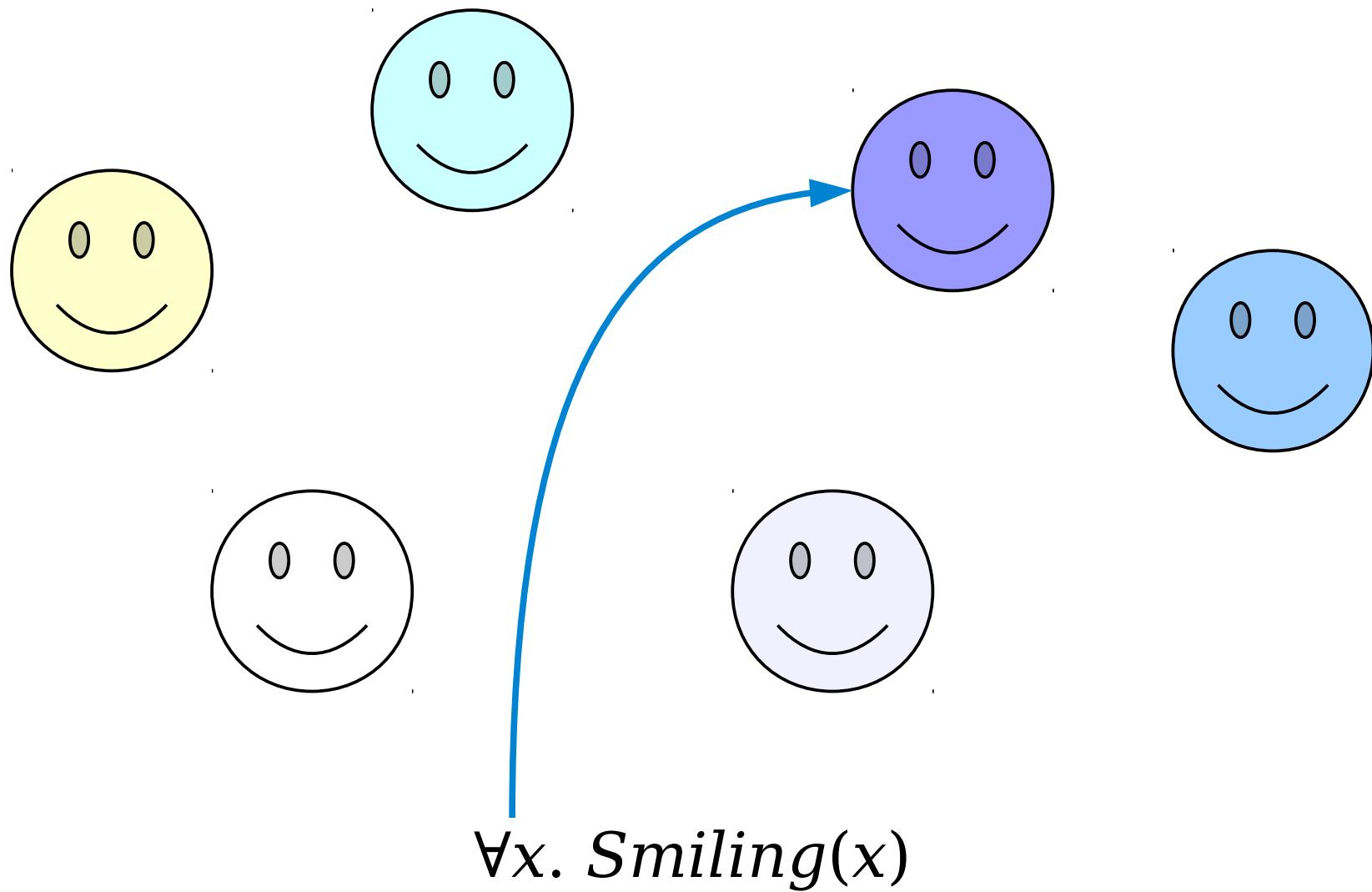
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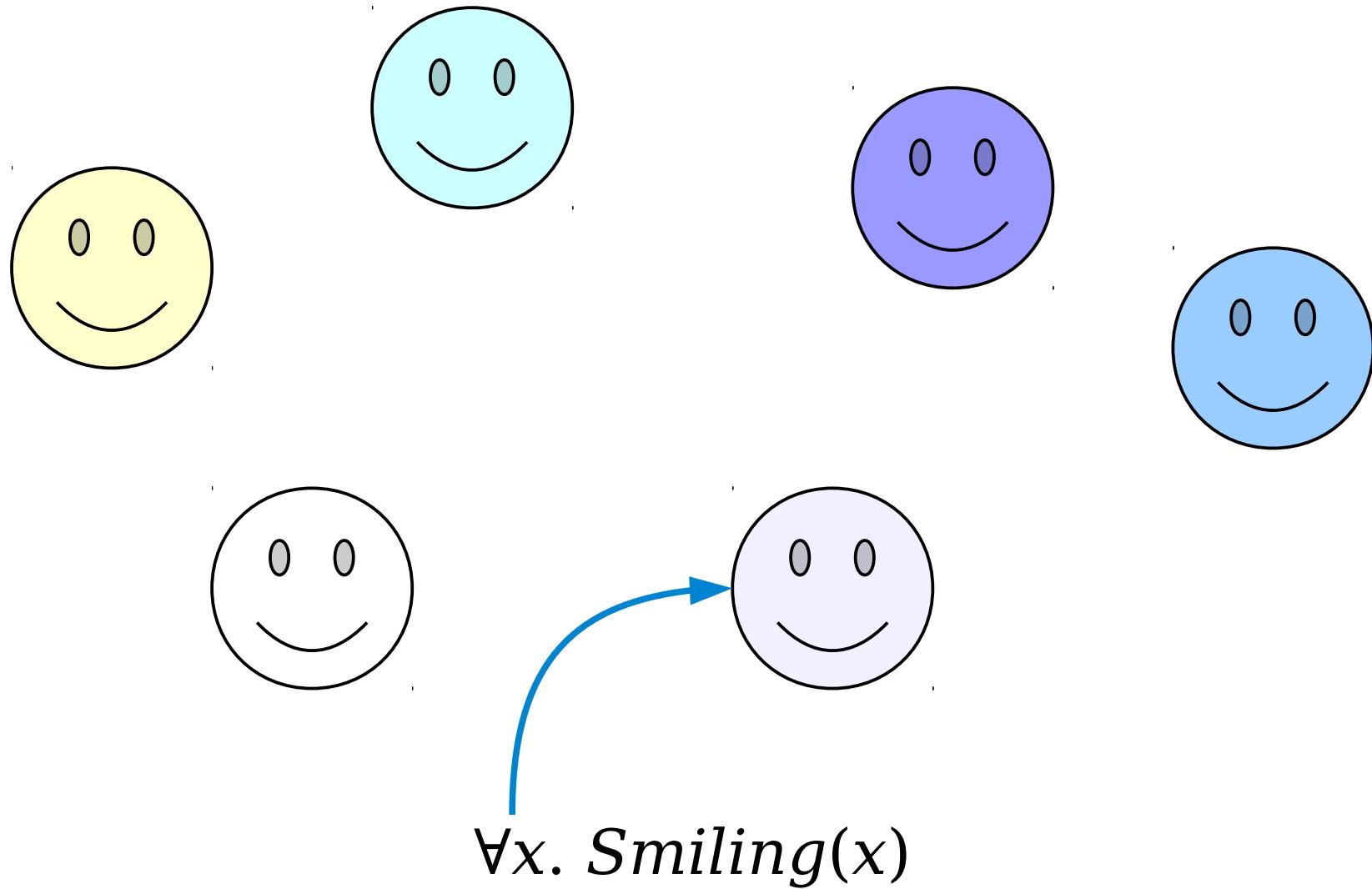
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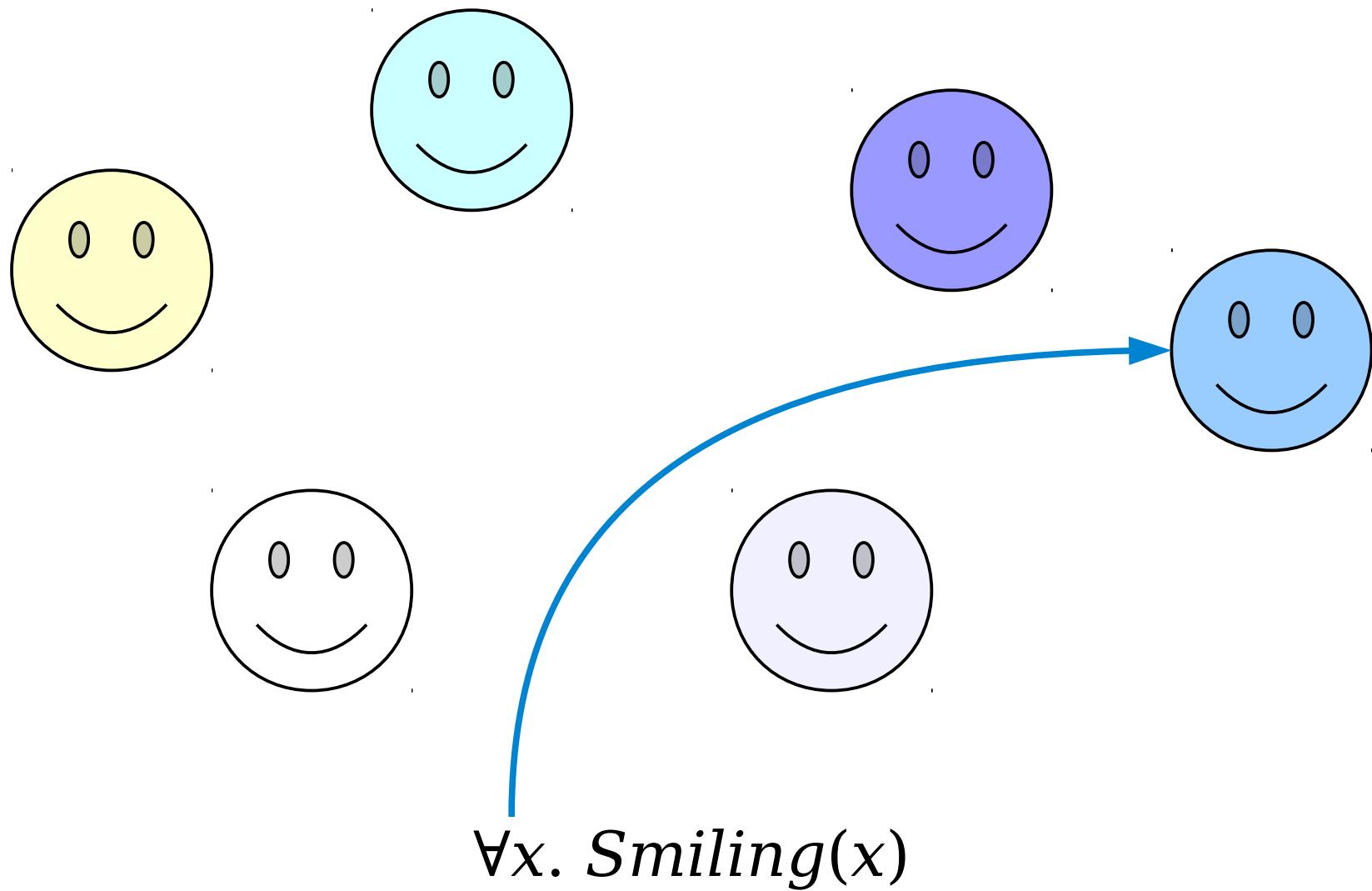
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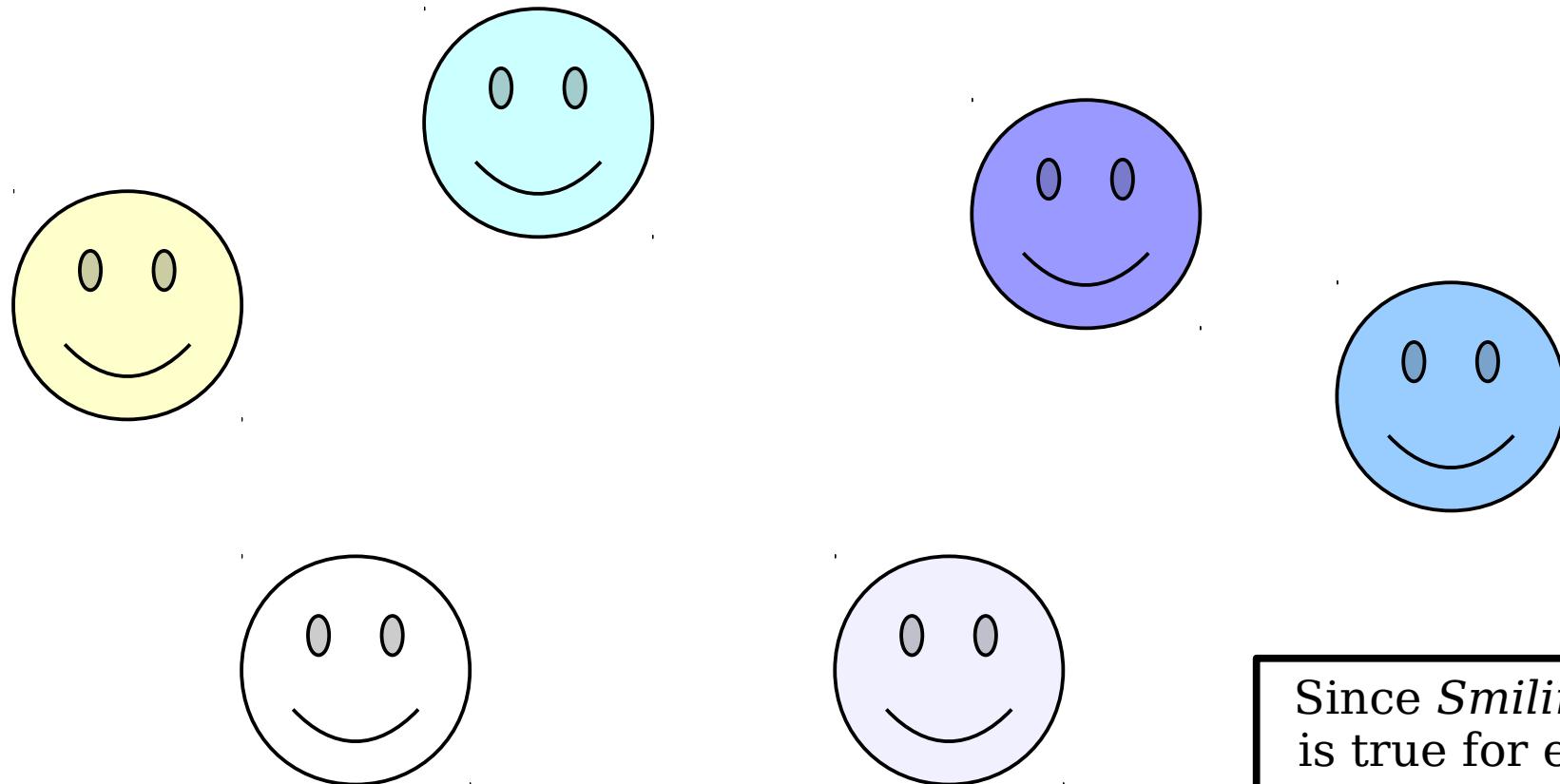
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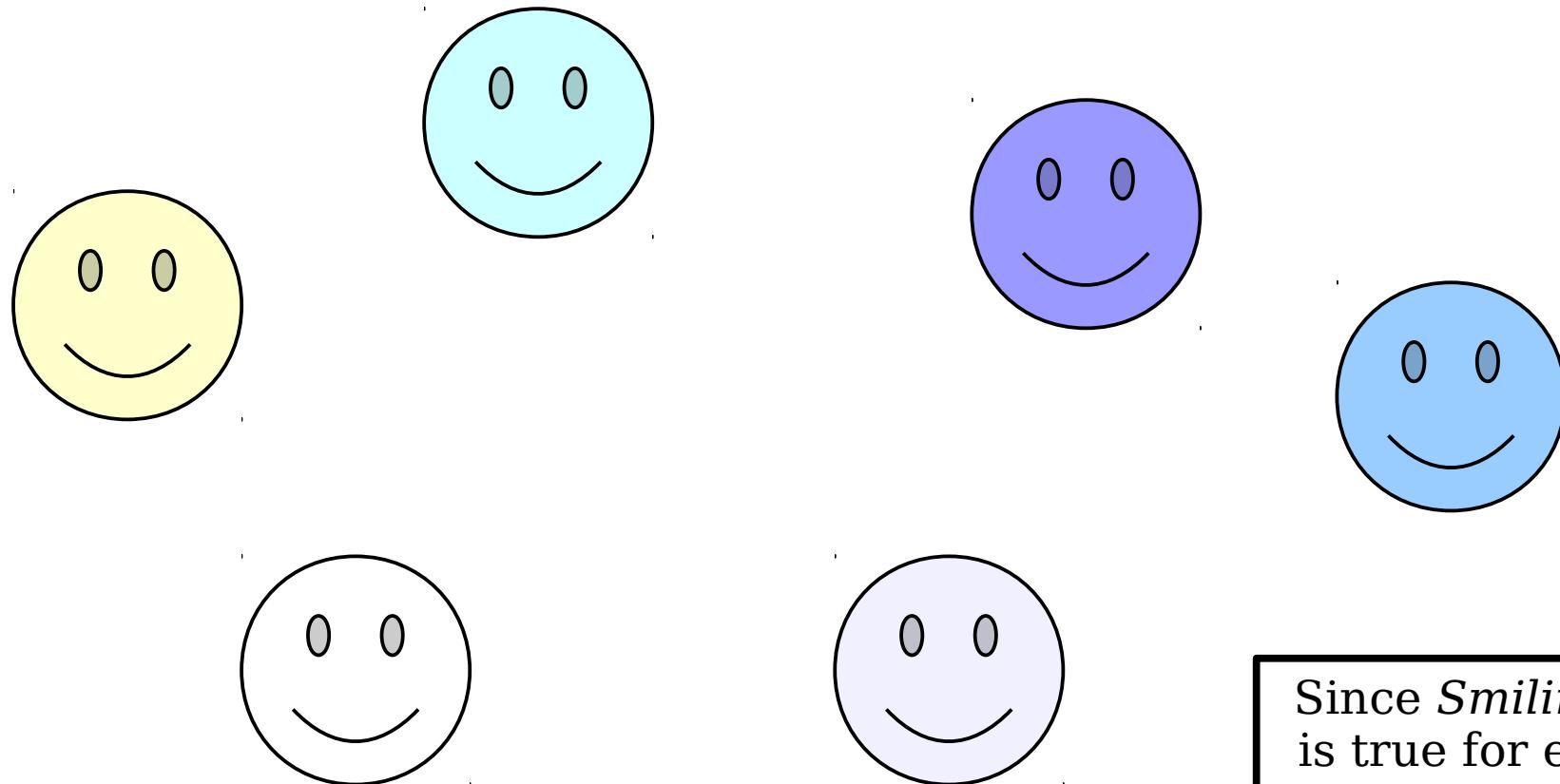


# The Universal Quantifier


$$\forall x. \text{Smiling}(x)$$

Since *Smiling*( $x$ ) is true for every choice of  $x$ , this statement evaluates to true.

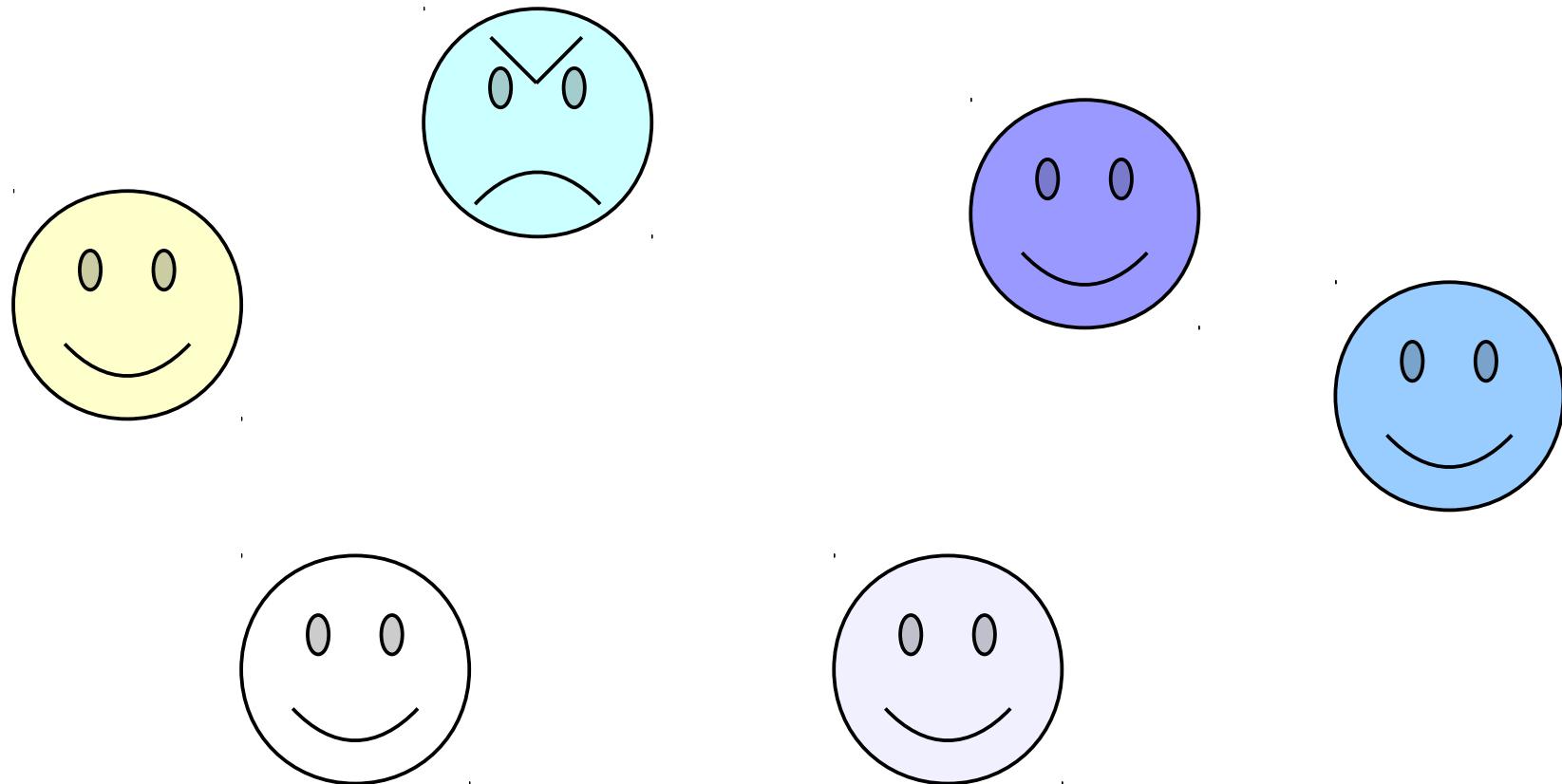
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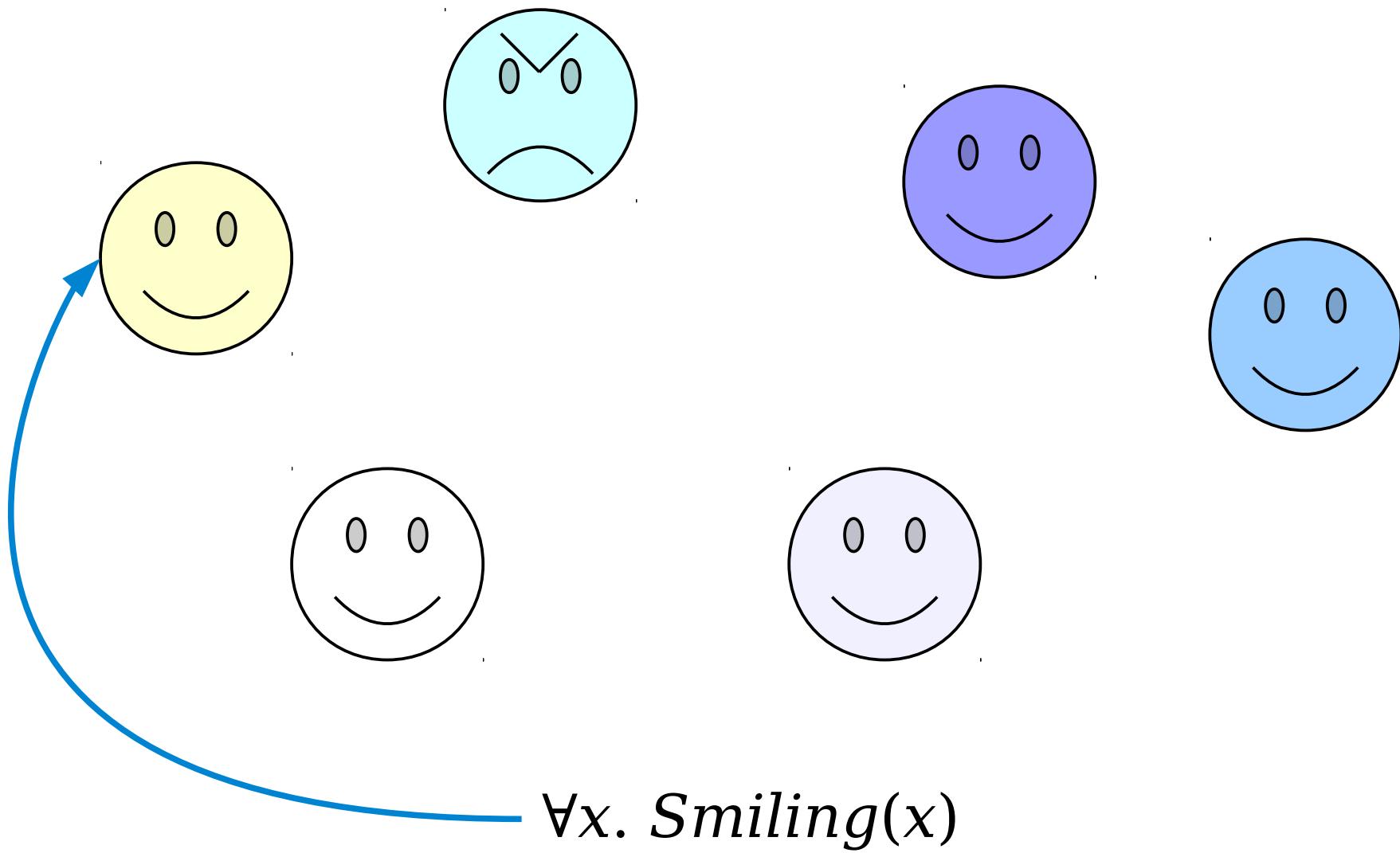
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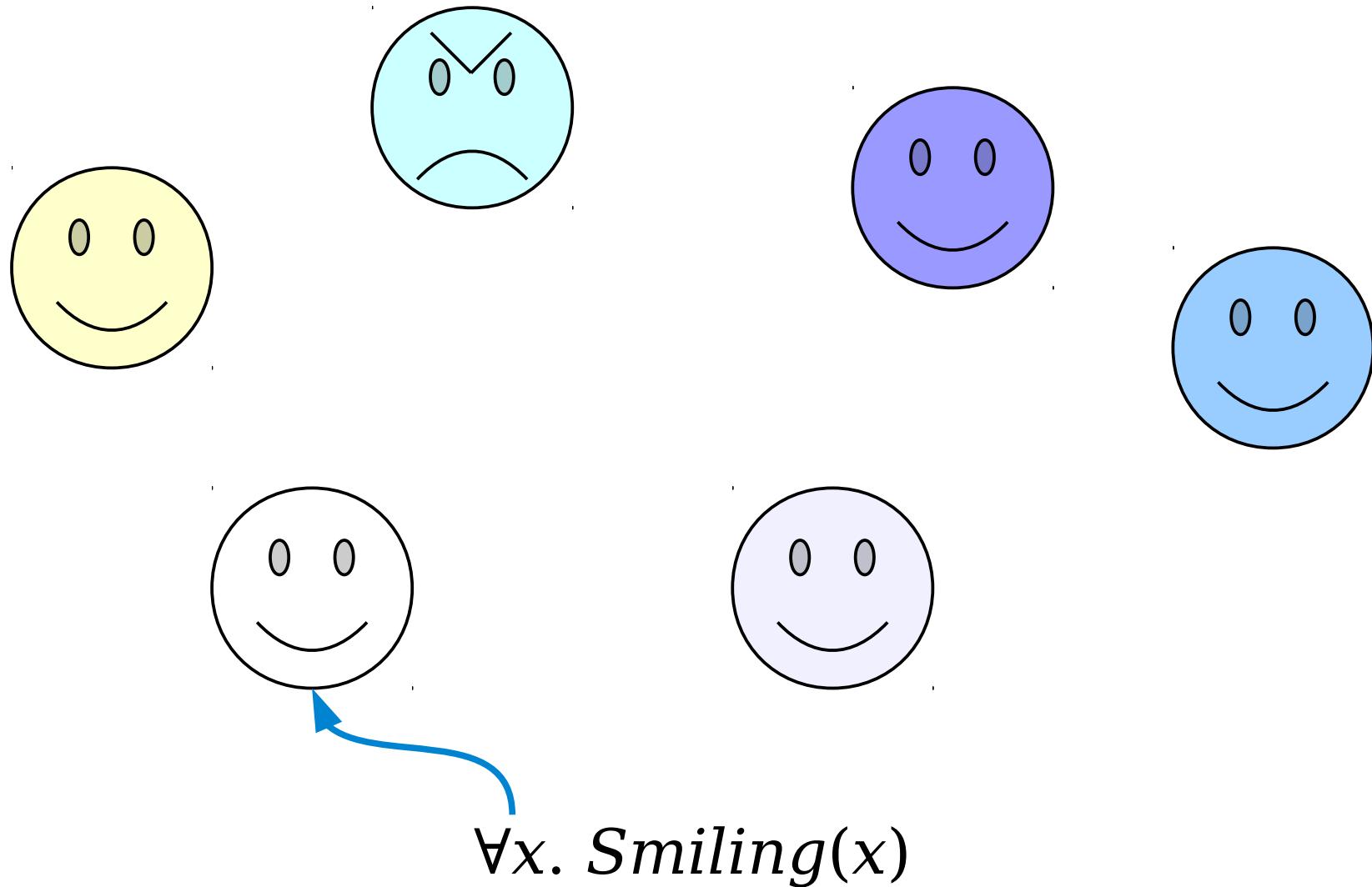
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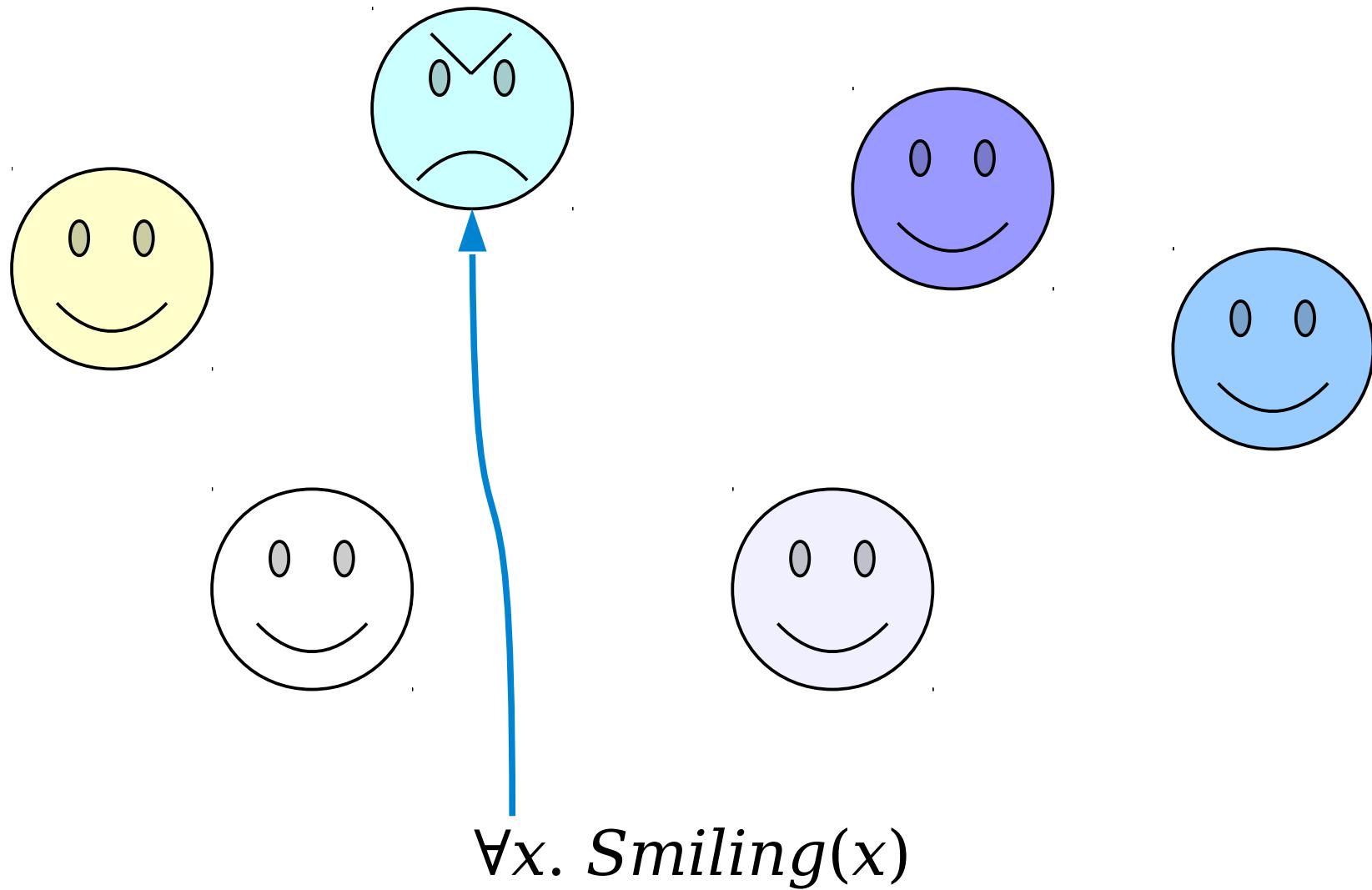
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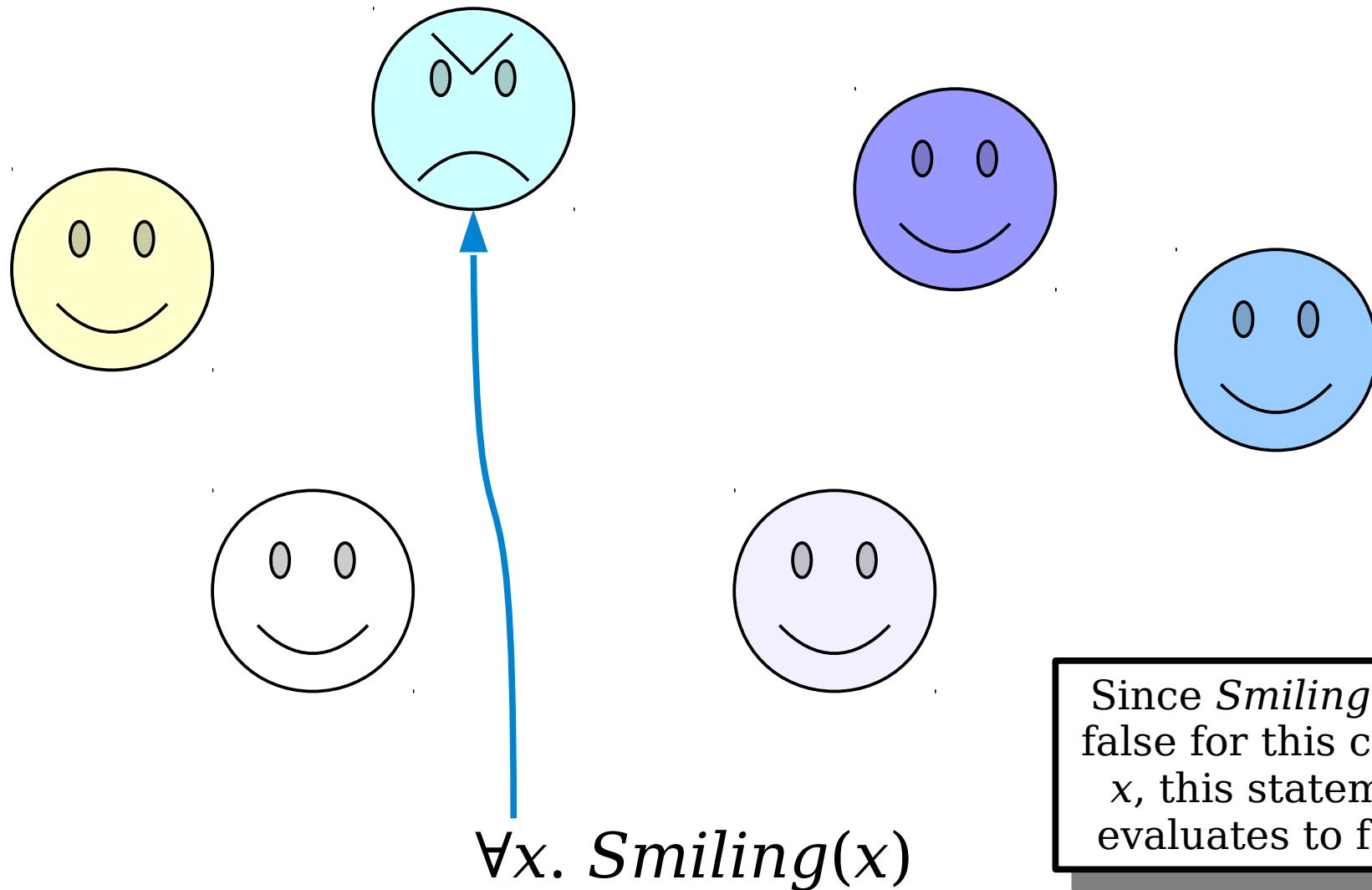
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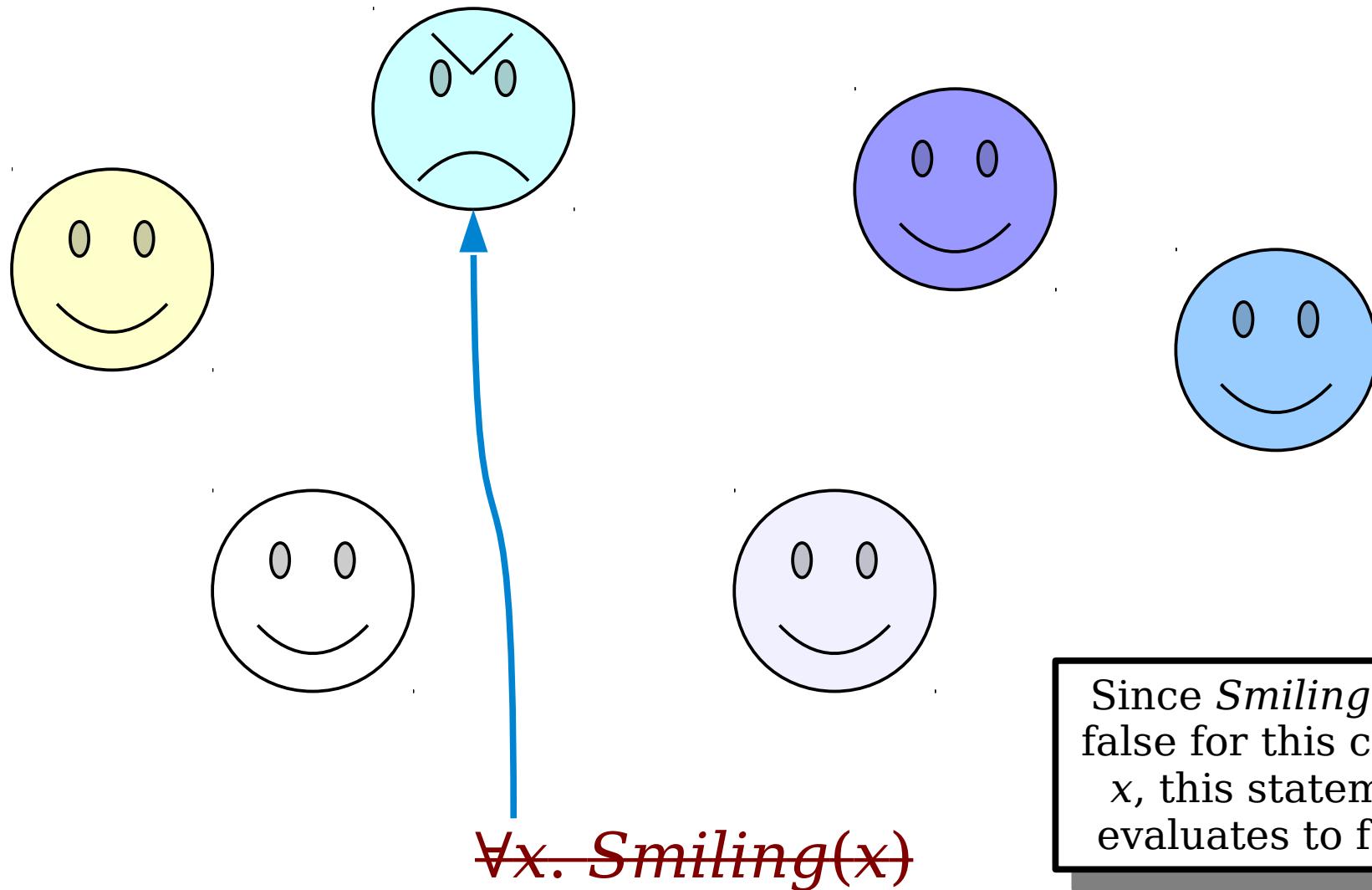
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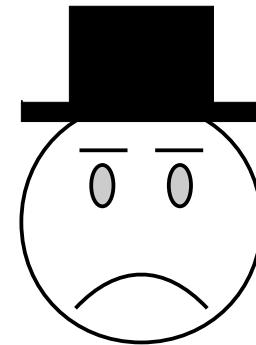
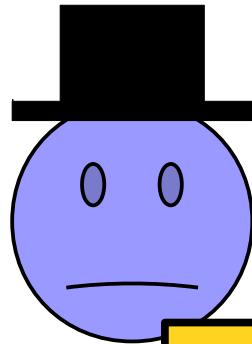
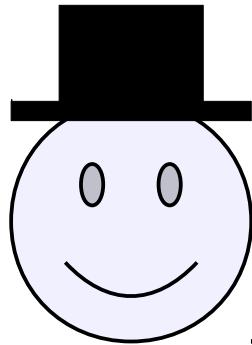
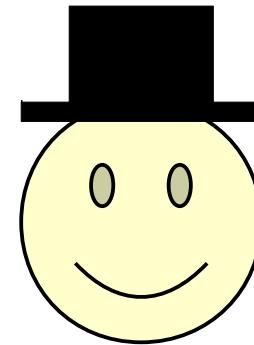
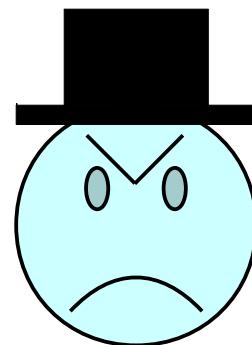
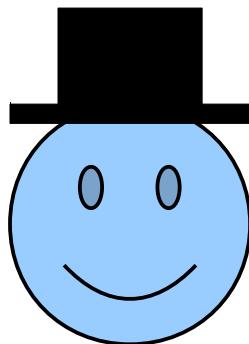
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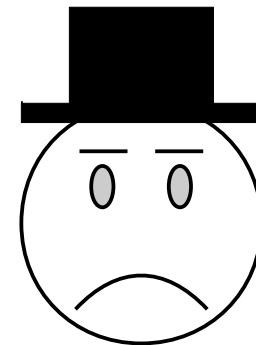
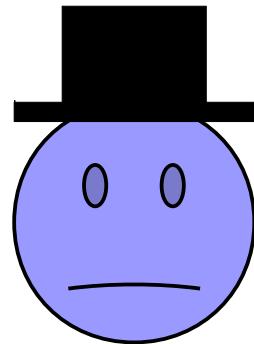
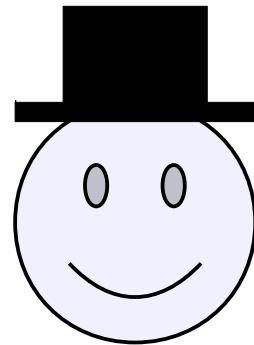
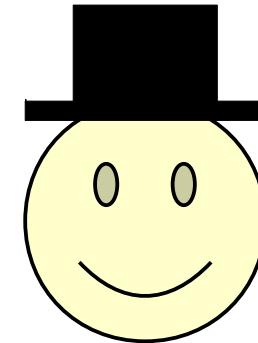
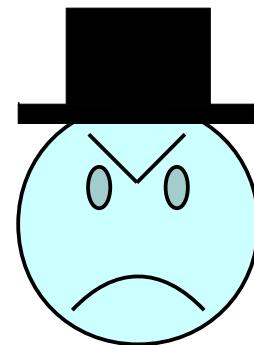
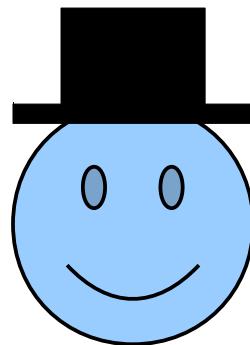
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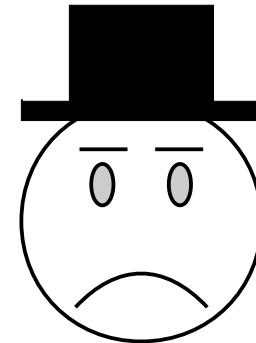
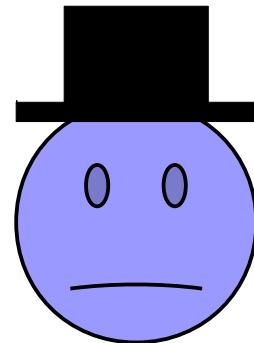
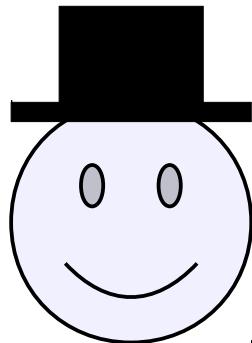
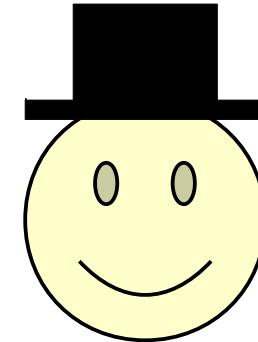
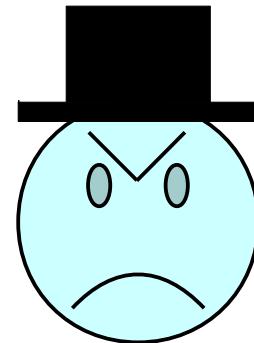
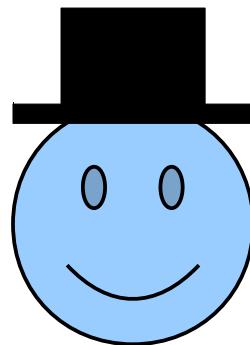
Answer at  
<https://cs103.stanford.edu/pollev>

$$(\forall x. Smiling(x)) \rightarrow (\forall y. WearingHat(y))$$

# The Universal Quantifier


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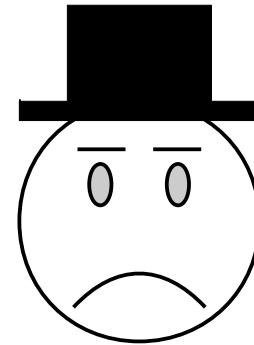
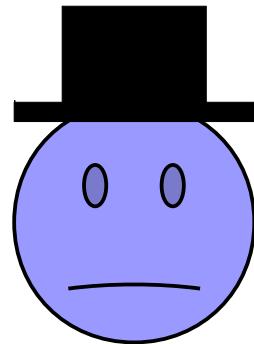
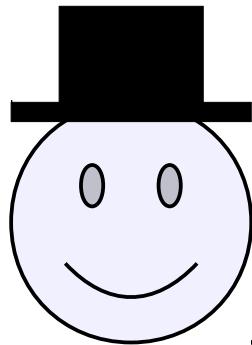
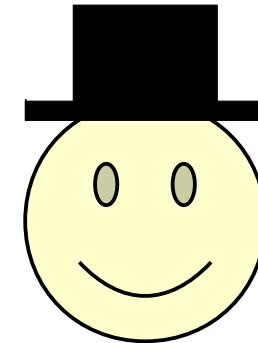
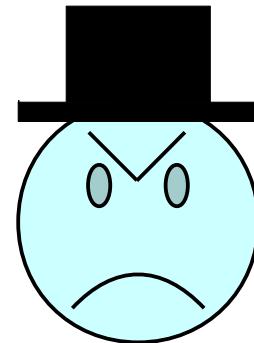
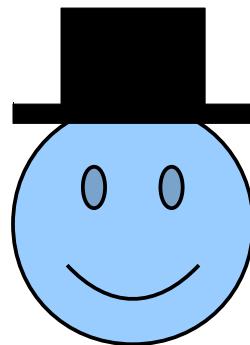
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Is this part of the statement true or false?

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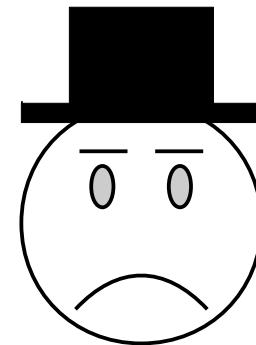
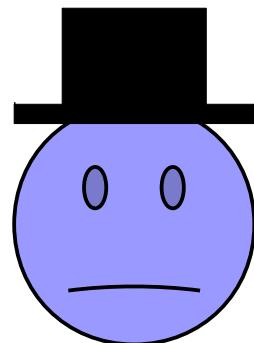
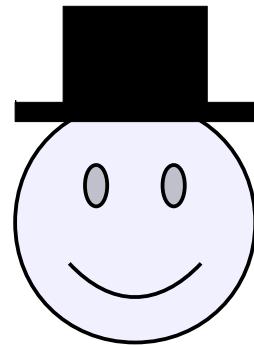
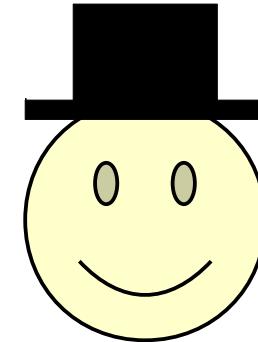
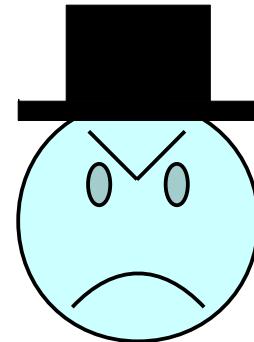
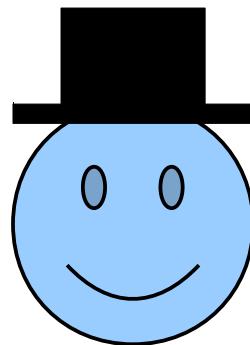
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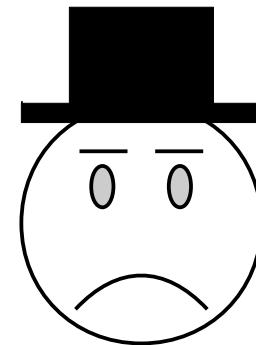
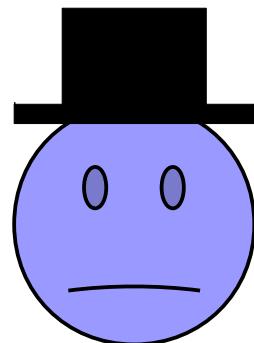
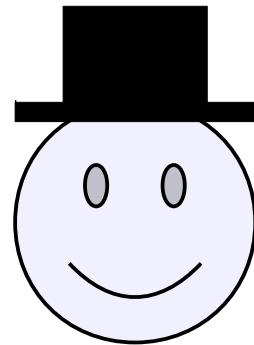
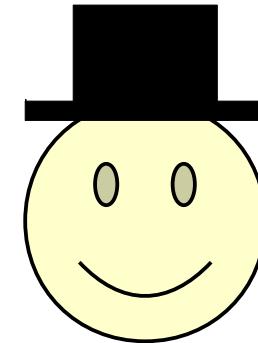
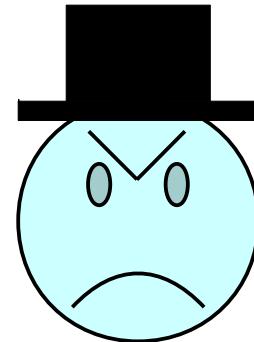
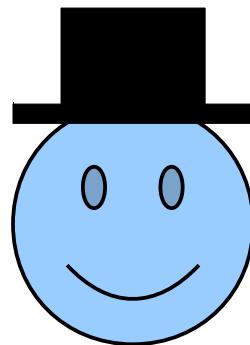
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Is this part of the statement true or false?

$$(\forall x. Smiling(x)) \rightarrow (\forall y. WearingHat(y))$$

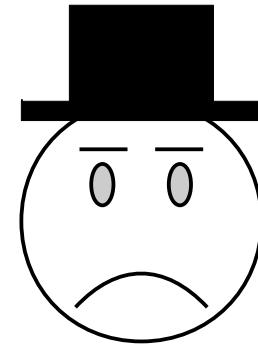
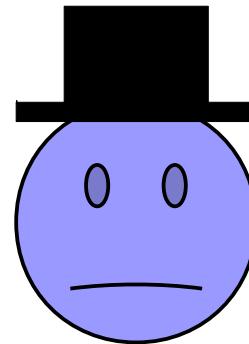
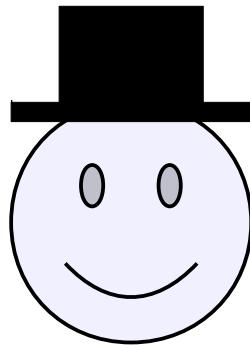
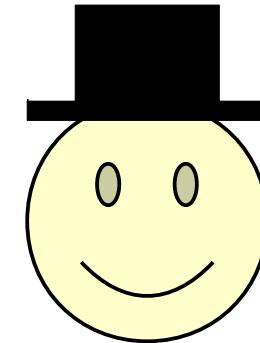
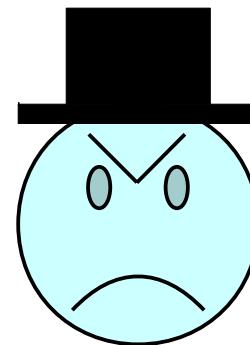
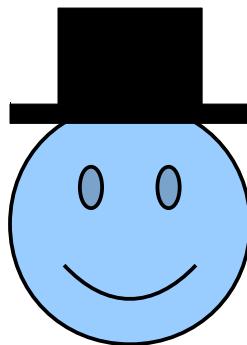
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Is this part of the statement true or false?

$(\forall x. \text{Smiling}(x)) \rightarrow (\forall y. \text{WearingHat}(y))$

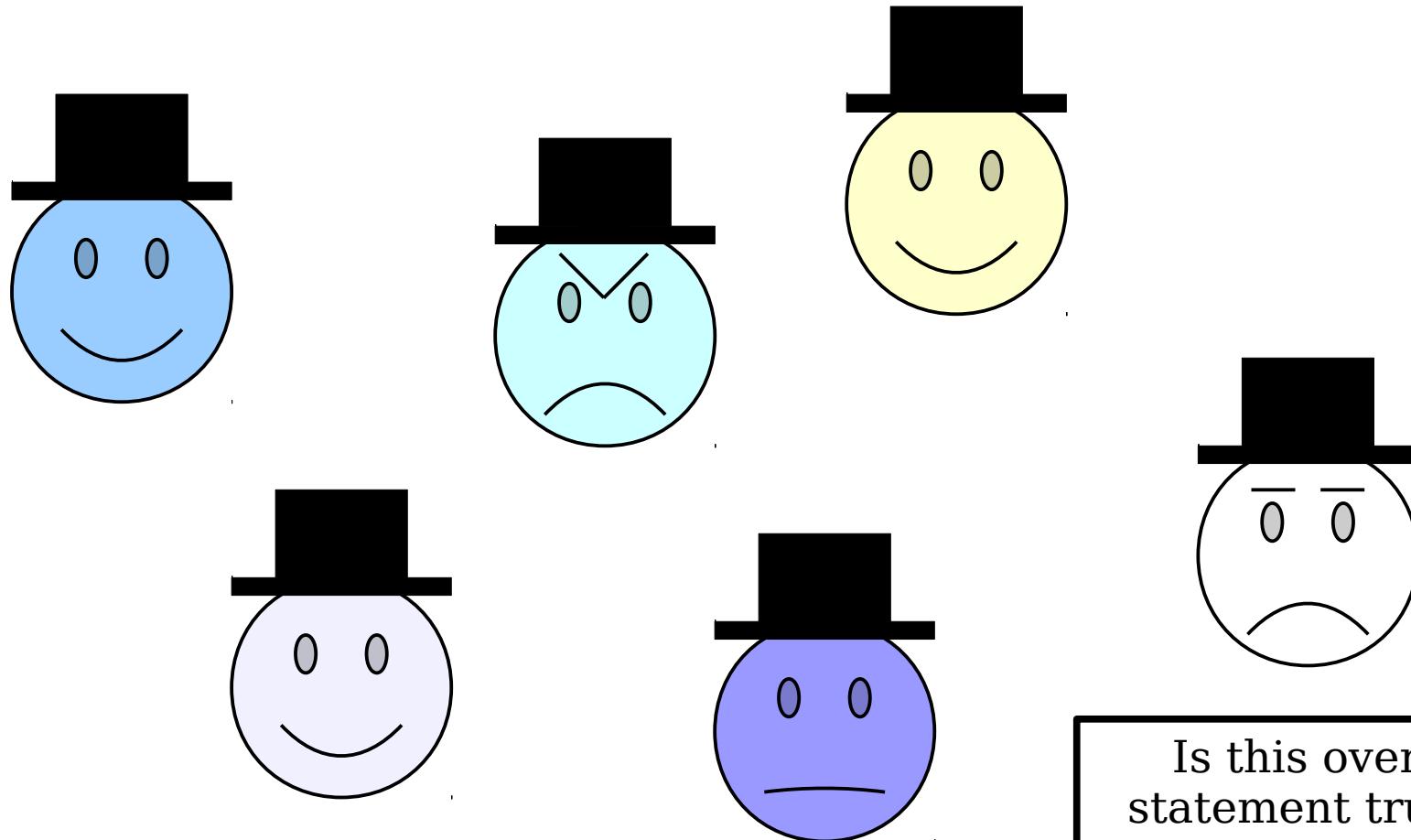
# The Universal Quantifier



Is this overall statement true or false in this scenario?

$(\forall x. \text{Smiling}(x)) \rightarrow (\forall y. \text{WearingHat}(y))$

# The Universal Quantifier



Is this overall statement true or false in this scenario?

$(\forall x. Smiling(x)) \rightarrow (\forall y. WearingHat(y))$

# Fun with Edge Cases

$\forall x. Smiling(x)$

# Fun with Edge Cases

Universally-quantified statements are said to be ***vacuously true*** in empty worlds.

$\forall x. Smiling(x)$

# Translating into First-Order Logic

# Translating Into Logic

- First-order logic is an excellent tool for manipulating definitions and theorems to learn more about them.
- Need to take a negation? Translate your statement into FOL, negate it, then translate it back.
- Want to prove something by contrapositive? Translate your implication into FOL, take the contrapositive, then translate it back.

# Translating Into Logic

- When translating from English into first-order logic, we recommend that you  
***think of first-order logic as a mathematical programming language.***
- Your goal is to learn how to combine basic concepts (quantifiers, connectives, etc.) together in ways that say what you mean.

## Using the predicates

- $Smiling(x)$ , which states that  $x$  is smiling, and
- $WearingHat(x)$ , which states that  $x$  is wearing a hat,

write a formula in first-order logic that says

***some smiling person wears a hat.***

How would you represent this in first-order logic?

Answer at

<https://cs103.stanford.edu/pollev>

## Using the predicates

- $\text{Smiling}(x)$ , which states that  $x$  is smiling, and
- $\text{WearingHat}(x)$ , which states that  $x$  is wearing a hat,

write a formula in first-order logic that says

***some smiling person wears a hat.***

Which of the following are correct translations?

- (A)  $\exists x. \text{WearingHat}(\text{Smiling}(x))$
- (B)  $\exists x. (\text{Smiling}(x) = \text{WearingHat}(x))$
- (C)  $\exists x. (\text{Smiling}(x) \wedge \text{WearingHat}(x))$
- (D)  $\exists x. (\text{Smiling}(x) \rightarrow \text{WearingHat}(x))$

## Using the predicates

- $Smiling(x)$ , which states that  $x$  is smiling, and
- $WearingHat(x)$ , which states that  $x$  is wearing a hat,

write a formula in first-order logic that says

***some smiling person wears a hat.***

Which of the following are correct translations?

- (A)  ~~$\exists x. WearingHat(Smilng(x))$~~
- (B)  $\exists x. (Smiling(x) = WearingHat(x))$
- (C)  $\exists x. (Smiling(x) \wedge WearingHat(x))$
- (D)  $\exists x. (Smiling(x) \rightarrow WearingHat(x))$

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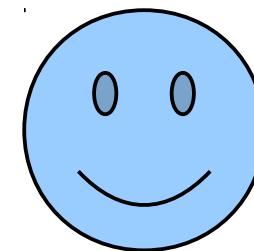
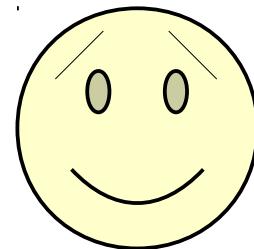
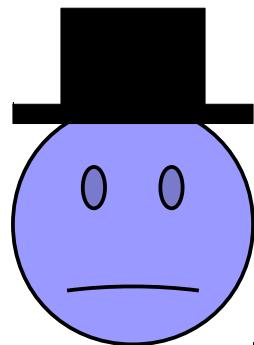
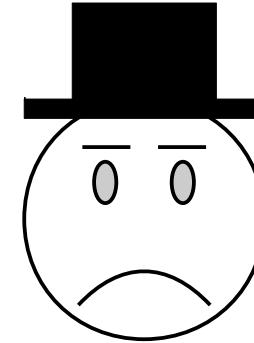
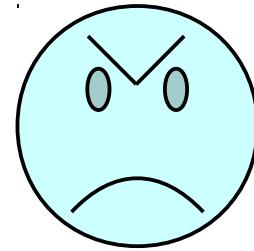
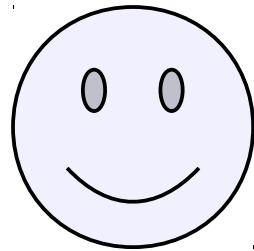
“Some smiling person wears a hat.”

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$\exists x. (Smiling(x) \wedge WearingHat(x))$

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$\exists x. (Smiling(x) \rightarrow WearingHat(x))$



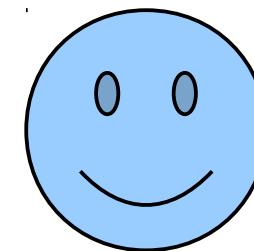
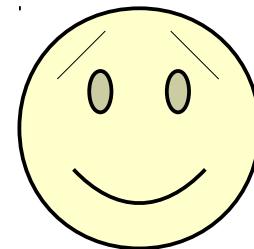
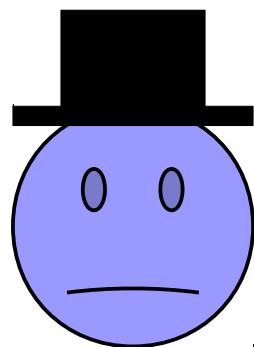
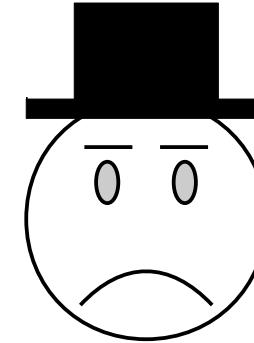
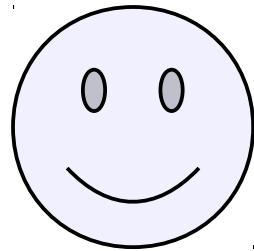
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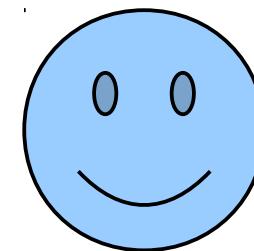
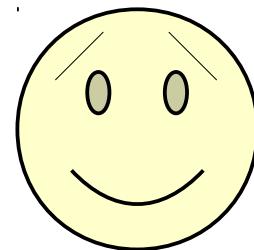
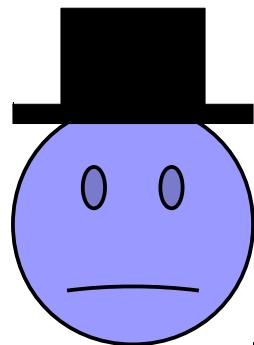
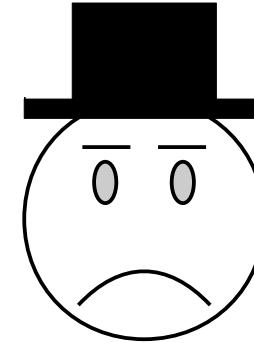
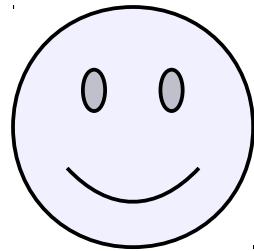
“Some smiling person wears a hat.”

---

$$\exists x. (Smiling(x) \wedge WearingHat(x))$$

---

$$\exists x. (Smiling(x) \rightarrow WearingHat(x))$$



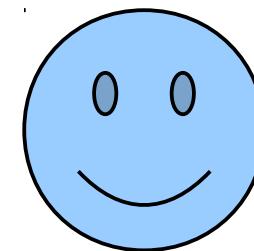
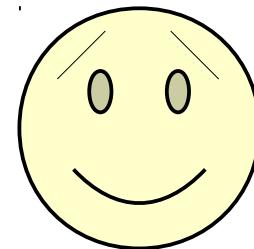
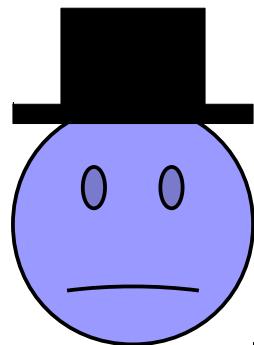
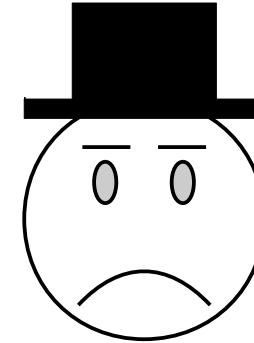
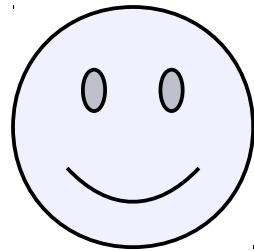
“Some smiling person wears a hat.” ***False***

---

$\exists x. (Smiling(x) \wedge WearingHat(x))$

---

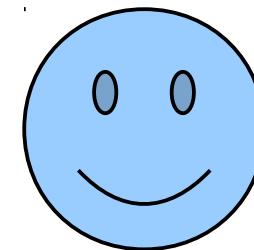
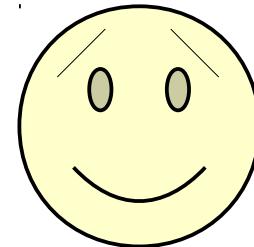
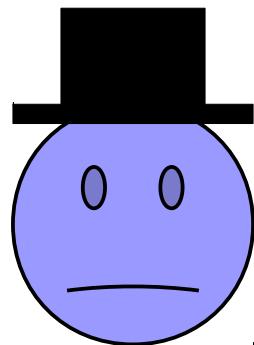
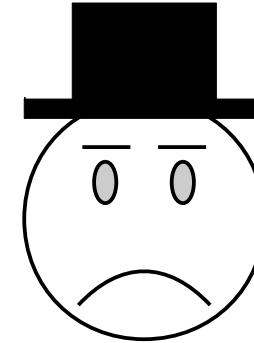
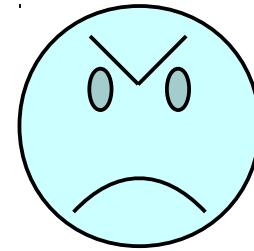
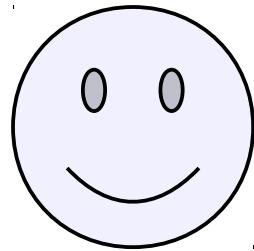
$\exists x. (Smiling(x) \rightarrow WearingHat(x))$



“Some smiling person wears a hat.” ***False***

$\exists x. (Smiling(x) \wedge WearingHat(x))$

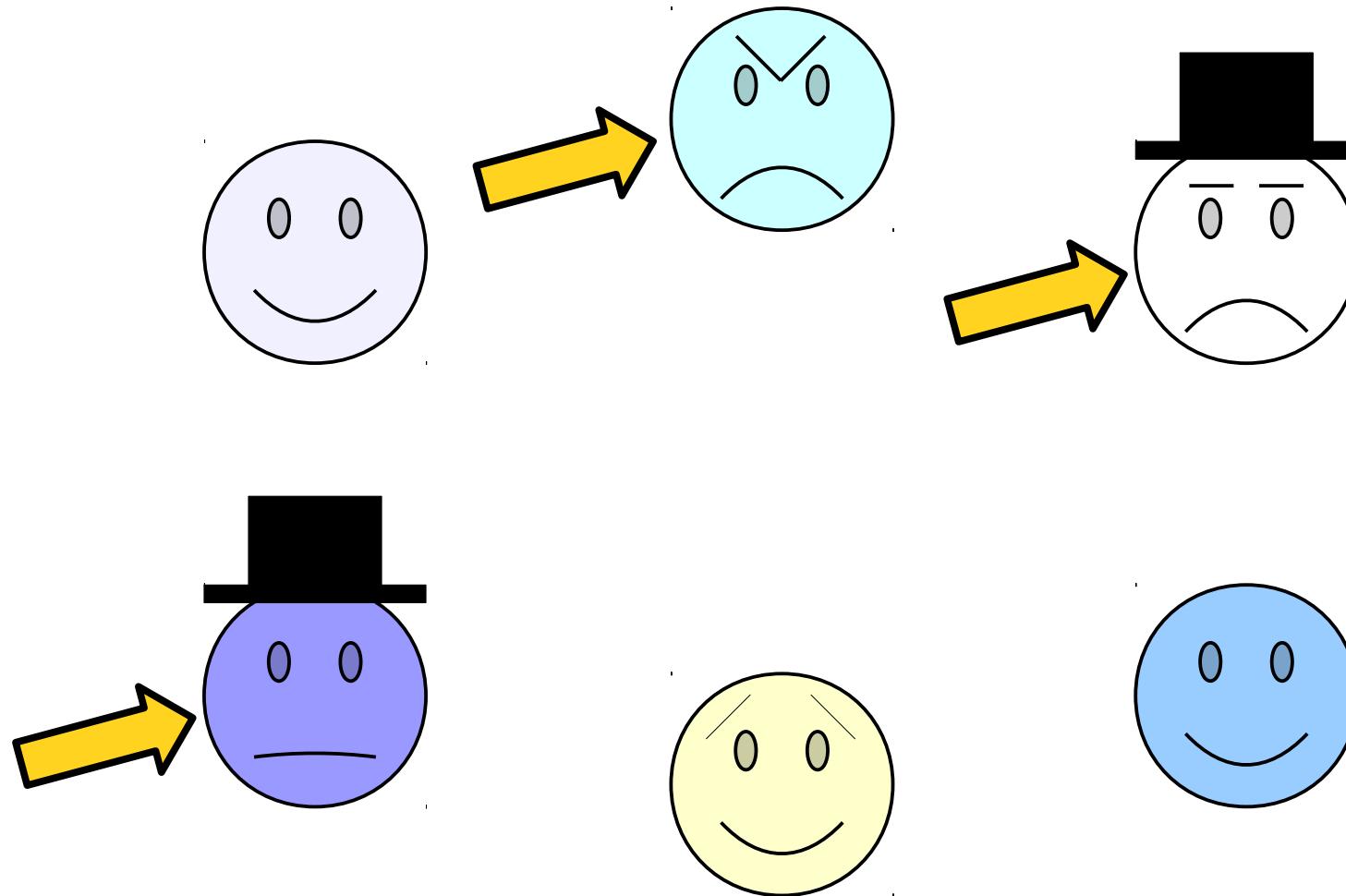
$\exists x. (Smiling(x) \rightarrow WearingHat(x))$



“Some smiling person wears a hat.” ***False***

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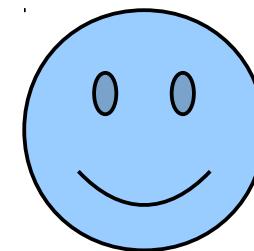
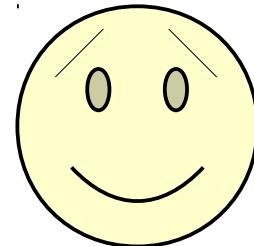
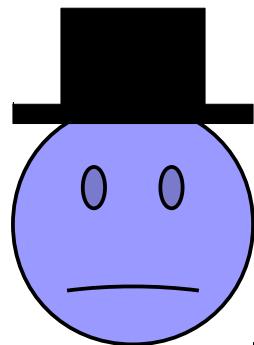
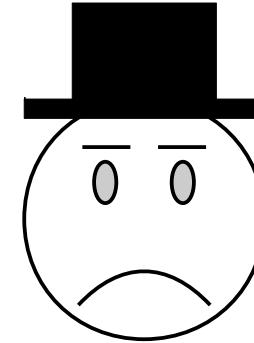
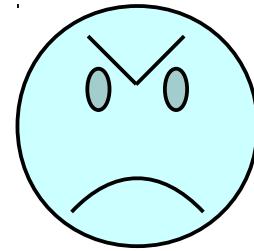
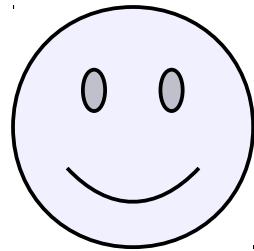


“Some smiling person wears a hat.” ***False***

---

$\exists x. (Smiling(x) \wedge WearingHat(x))$  ***False***

$\exists x. (Smiling(x) \rightarrow WearingHat(x))$  ***True***



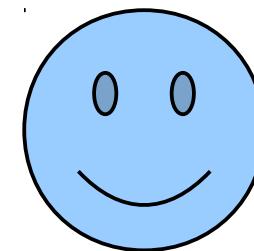
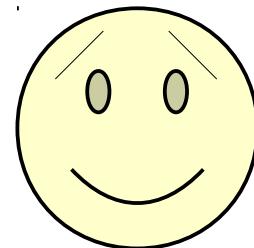
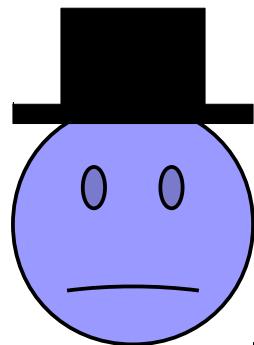
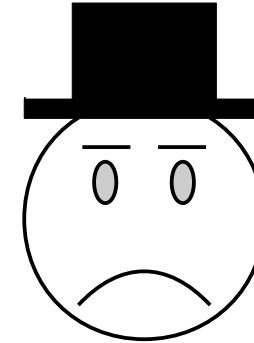
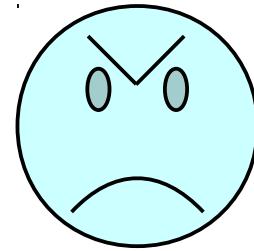
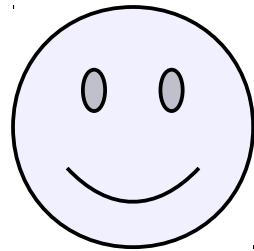
“Some smiling person wears a hat.” ***False***

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$\exists x. (Smiling(x) \wedge WearingHat(x))$  ***False***

---

$\exists x. (Smiling(x) \rightarrow WearingHat(x))$  ***True***



“Some smiling person wears a hat.” ***False***

---

$\exists x. (Smiling(x) \wedge WearingHat(x))$  ***False***

---

~~$\exists x. (Smiling(x) \rightarrow WearingHat(x))$~~  ***True***

“Some  $P$  is a  $Q$ ”

translates as

$\exists x. (P(x) \wedge Q(x))$

## *Useful Intuition:*

Existentially-quantified statements are false unless there's a positive example.

$$\exists x. (P(x) \wedge Q(x))$$

If  $x$  is an example, it must have property  $P$  on top of property  $Q$ .

## Using the predicates

- $Smiling(x)$ , which states that  $x$  is smiling, and
- $WearingHat(x)$ , which states that  $x$  is wearing a hat,

write a sentence in first-order logic that says

***every smiling person wears a hat.***

Which of the following are correct translations?

- (A)  $\forall x. (Smiling(x) \wedge WearingHat(x))$
- (B)  $\forall x. (Smiling(x) \rightarrow WearingHat(x))$

Answer at

<https://cs103.stanford.edu/pollev>

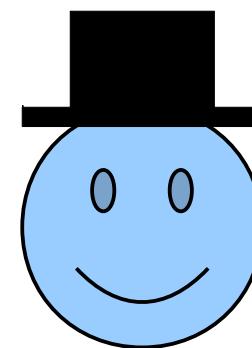
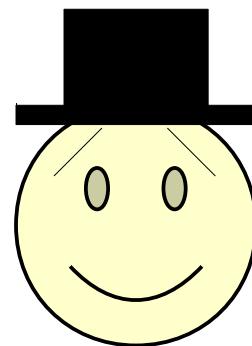
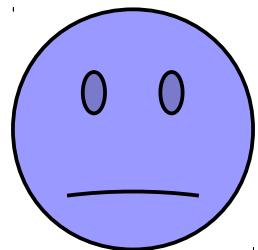
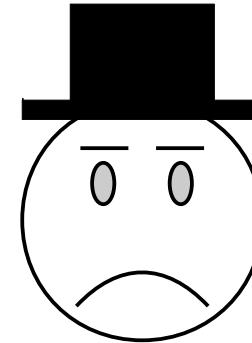
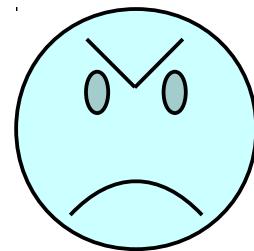
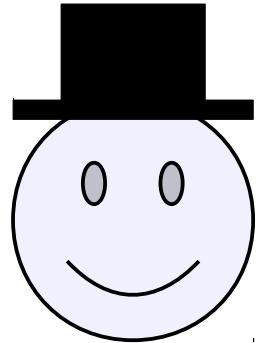
“Every smiling person wears a hat.”

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$\forall x. (Smiling(x) \wedge WearingHat(x))$

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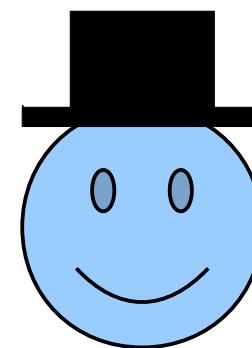
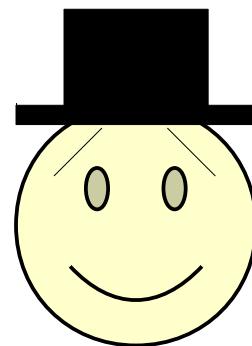
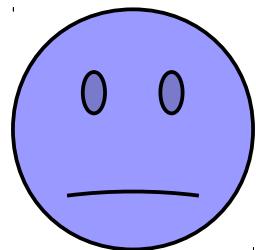
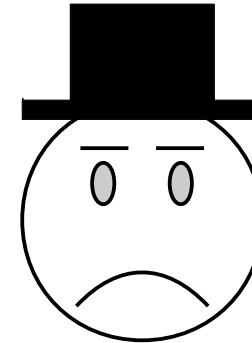
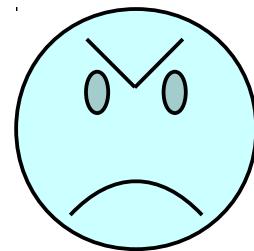
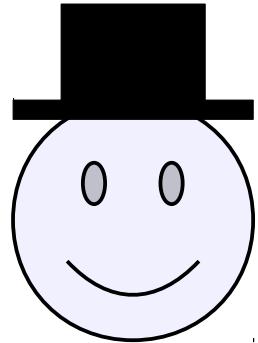
“Every smiling person wears a hat.”

---

$$\forall x. (Smiling(x) \wedge WearingHat(x))$$

---

$$\forall x. (Smiling(x) \rightarrow WearingHat(x))$$



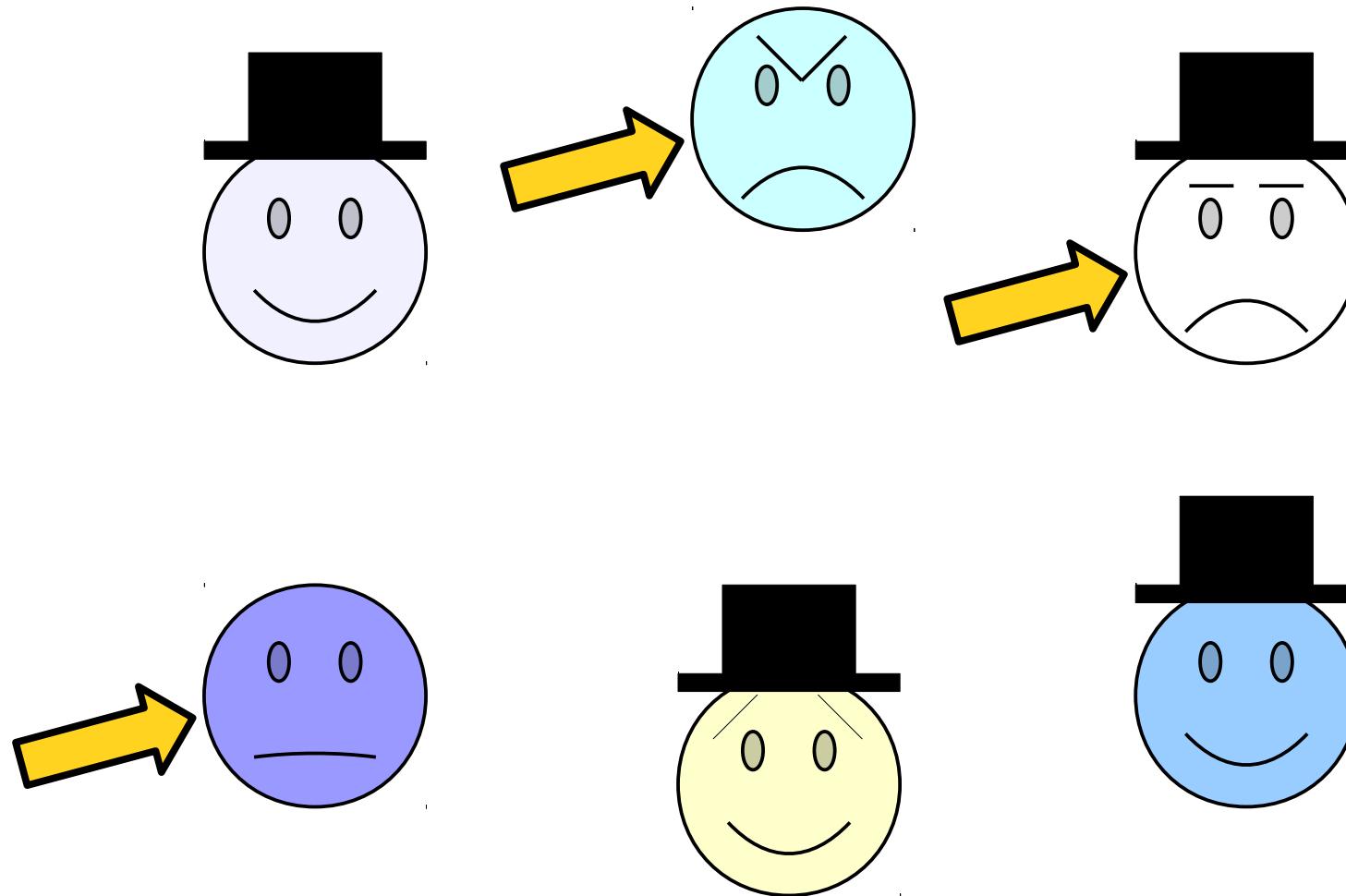
“Every smiling person wears a hat.” **True**

---

$$\forall x. (Smiling(x) \wedge WearingHat(x))$$

---

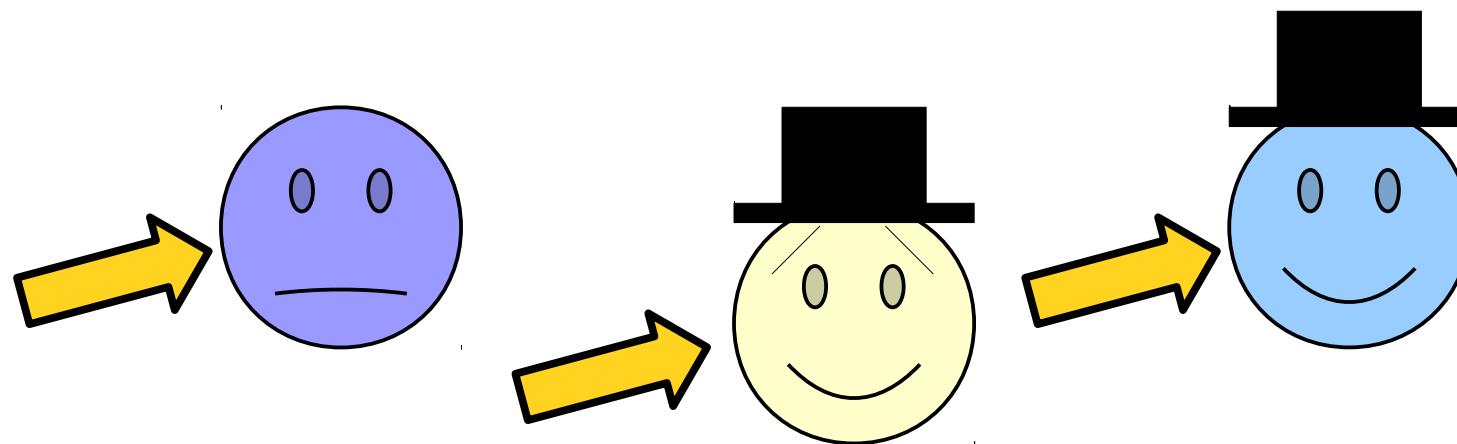
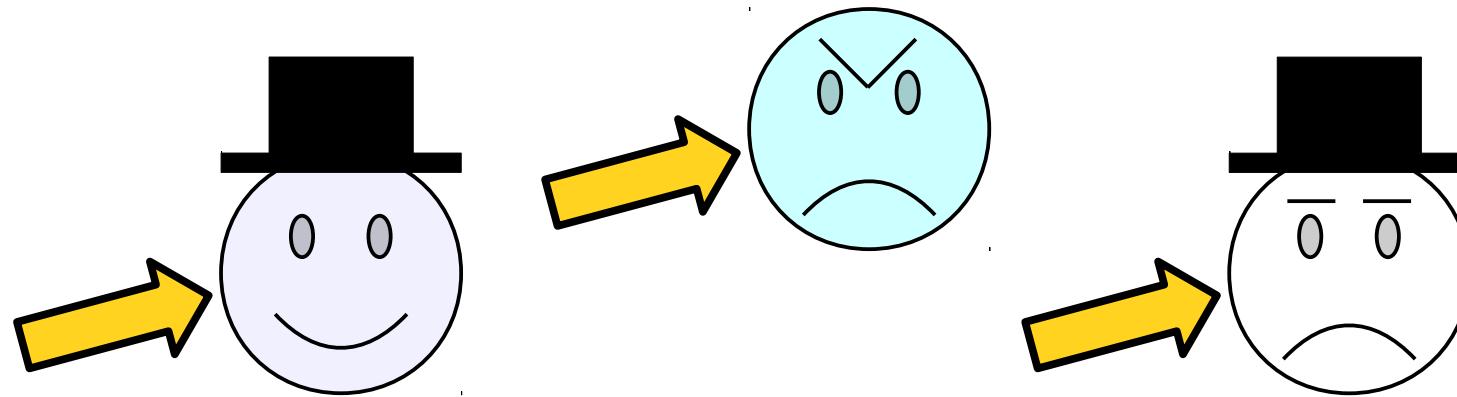
$$\forall x. (Smiling(x) \rightarrow WearingHat(x))$$



“Every smiling person wears a hat.” **True**

$\forall x. (Smiling(x) \wedge WearingHat(x))$  **False**

$\forall x. (Smiling(x) \rightarrow WearingHat(x))$

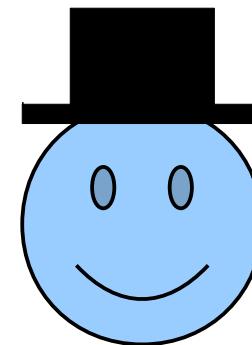
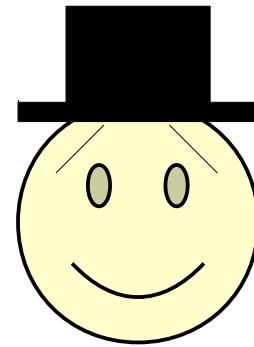
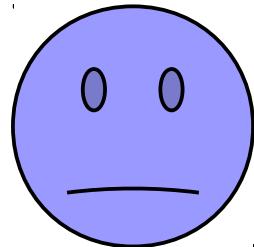
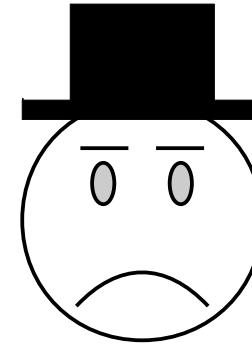
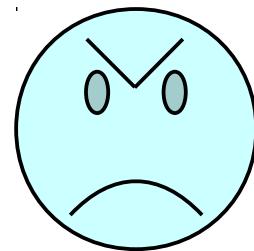
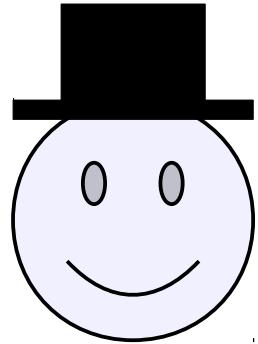


“Every smiling person wears a hat.” **True**

---

$\forall x. (Smiling(x) \wedge WearingHat(x))$  **False**

$\forall x. (Smiling(x) \rightarrow WearingHat(x))$  **True**



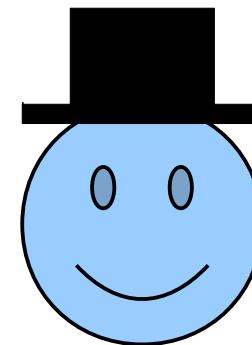
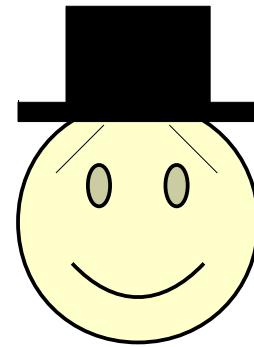
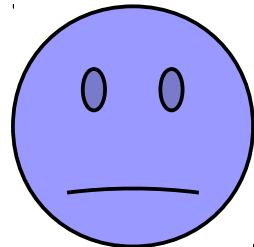
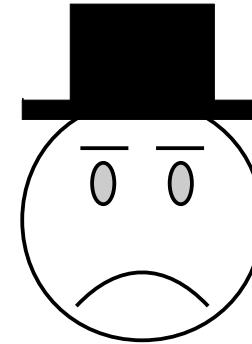
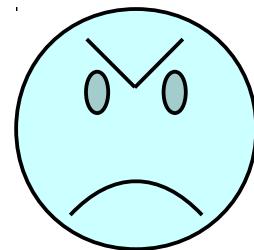
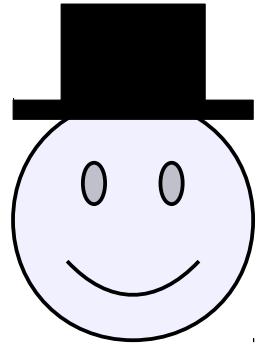
“Every smiling person wears a hat.” **True**

---

$\forall x. (Smiling(x) \wedge WearingHat(x))$  **False**

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$\forall x. (Smiling(x) \rightarrow WearingHat(x))$  **True**



“Every smiling person wears a hat.” **True**

~~$\forall x. (Smiling(x) \wedge WearingHat(x))$~~  **False**

$\forall x. (Smiling(x) \rightarrow WearingHat(x))$  **True**

“All  $P$ 's are  $Q$ 's”

translates as

$\forall x. (P(x) \rightarrow Q(x))$

## *Useful Intuition:*

Universally-quantified statements are true unless there's a counterexample.

$$\forall x. (P(x) \rightarrow Q(x))$$

If  $x$  is a counterexample, it must have property  $P$  but not have property  $Q$ .

# Good Pairings

- The  $\forall$  quantifier *usually* is paired with  $\rightarrow$ .

$$\forall x. (P(x) \rightarrow Q(x))$$

- The  $\exists$  quantifier *usually* is paired with  $\wedge$ .

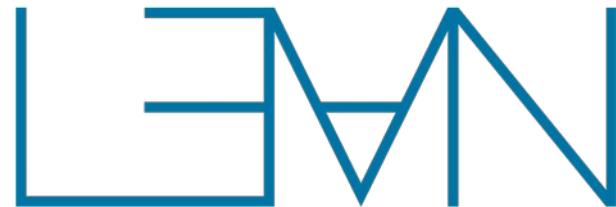
$$\exists x. (P(x) \wedge Q(x))$$

- In the case of  $\forall$ , the  $\rightarrow$  connective prevents the statement from being *false* when speaking about some object you don't care about.
- In the case of  $\exists$ , the  $\wedge$  connective prevents the statement from being *true* when speaking about some object you don't care about.

# Quantifiers in the Wild



Center for Automated Reasoning at Stanford University



theorem prover

# Next Time

- ***First-Order Translations***
  - How do we translate from English into first-order logic?
- ***Quantifier Orderings***
  - How do you select the order of quantifiers in first-order logic formulas?
- ***Negating Formulas***
  - How do you mechanically determine the negation of a first-order formula?
- ***Expressing Uniqueness***
  - How do we say there's just one object of a certain type?